1 Creagh-Whelan potential

1.1 Hamiltonian

A slightly extended version of the Hamiltonian used in Ref. [1] is studied here:

$$H^{(\kappa)} = T + V = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \left(x^2 - 1\right)^\kappa + Ax + Bxy^2 + Cx^2y^2 + \mu y^2, \tag{1}$$

where $\kappa = 2, 4$ (denoted as quadratic, quartic case, respectively) and A, B, C, μ are adjustable parameters:

- A is the control parameter of the phase transition. The phase transition occurs at $A = 0$, spinodal and antispinodal points are at $A = \pm\sqrt{64/27} \approx \pm 1.540$ ($A \approx \pm 1.904$) for $\kappa = 2$ ($\kappa = 4$). The dependence of the potential on the parameter A is shown in Fig. 3.
- B is the asymmetry parameter that squeezes one well and extends the other well along the y axis (see the fourth column in Figs. 1–2).
- C squeezes symmetrically both wells along the y axis (see the second column in Figs. 1–2).
- μ determines the chaoticity. It is observed in the Poincaré sections (see the first $\mu = 0.2$ and the third $\mu = 1$ column of Figs. 1–2) that the smaller positive value of μ the more chaotic the system is.

1.2 Classical dynamics

Classical dynamics and chaoticity of the system is demonstrated using the Poincaré sections for $A = 0$ and for various values of the parameters B, C, μ in Fig. 1 ($\kappa = 2$) and Fig. 2 ($\kappa = 4$), and for nonzero $A = 1, 2$ and $B = 0, C = 1, \mu = 0.2$ in Fig. 3.

1.3 Rigidity for $\kappa = 2$

At the point $A = 0$ of the phase transition, both left $x_0 = -1$ and right $x_0 = 1$ minima have the same depth $\bar{V}(x_0 = \pm 1, y = 0) = 0$ and they are separated by the barrier $\bar{V}(x = 0, y = 0) = 1$. The Taylor expansion at the minima $x = \pm 1$ in the x-direction is

$$\begin{aligned} \bar{V}(x, y = 0)|_{x_0=-1} &= -A + A(x+1) + 4(x+1)^2 + O(x+1)^3, \\ \bar{V}(x, y = 0)|_{x_0=1} &= A + A(x-1) + 4(x-1)^2 + O(x-1)^3, \end{aligned} \tag{2}$$

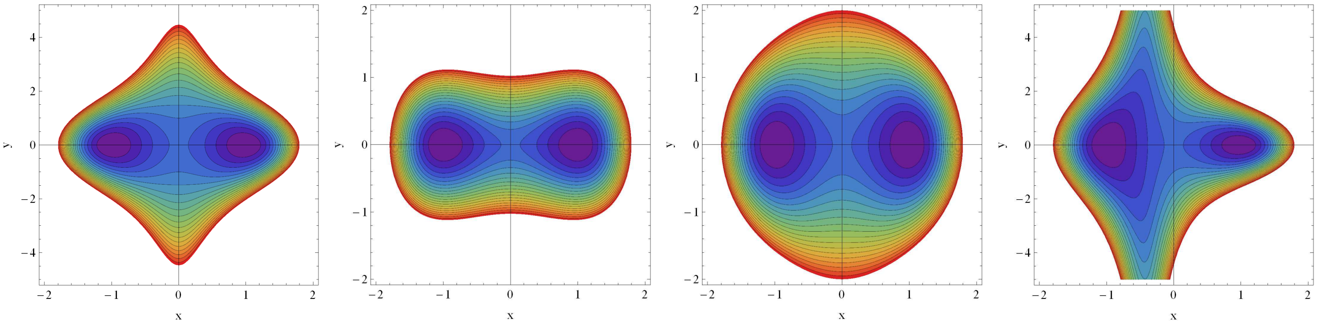
and in the y-direction around $y = 0$

$$\begin{aligned} \bar{V}(x_0 = -1, y) &= -A + (-B + C + \mu)y^2 + O(y)^3, \\ \bar{V}(x_0 = 1, y) &= A + (B + C + \mu)^2 + O(y)^3. \end{aligned} \tag{3}$$

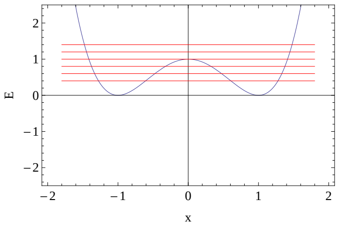
By comparing the quadratic term with the potential of the harmonic oscillator $V(x) = \Omega^2 x^2/2$, the frequencies of small x, y vibrations are

$$\begin{aligned} \bar{\Omega}_x^{(\pm)} &= 2\sqrt{2}, \\ \bar{\Omega}_y^{(-)} &= \sqrt{2(-B + C + \mu)}, \\ \bar{\Omega}_y^{(+)} &= \sqrt{2(B + C + \mu)}. \end{aligned} \tag{4}$$

These expressions imply the following classification of the rigidity of both minima:



(a) $A = 0$ (b) $C = 3.8$ (c) $\mu = 1$ (d) $B = 0.8$



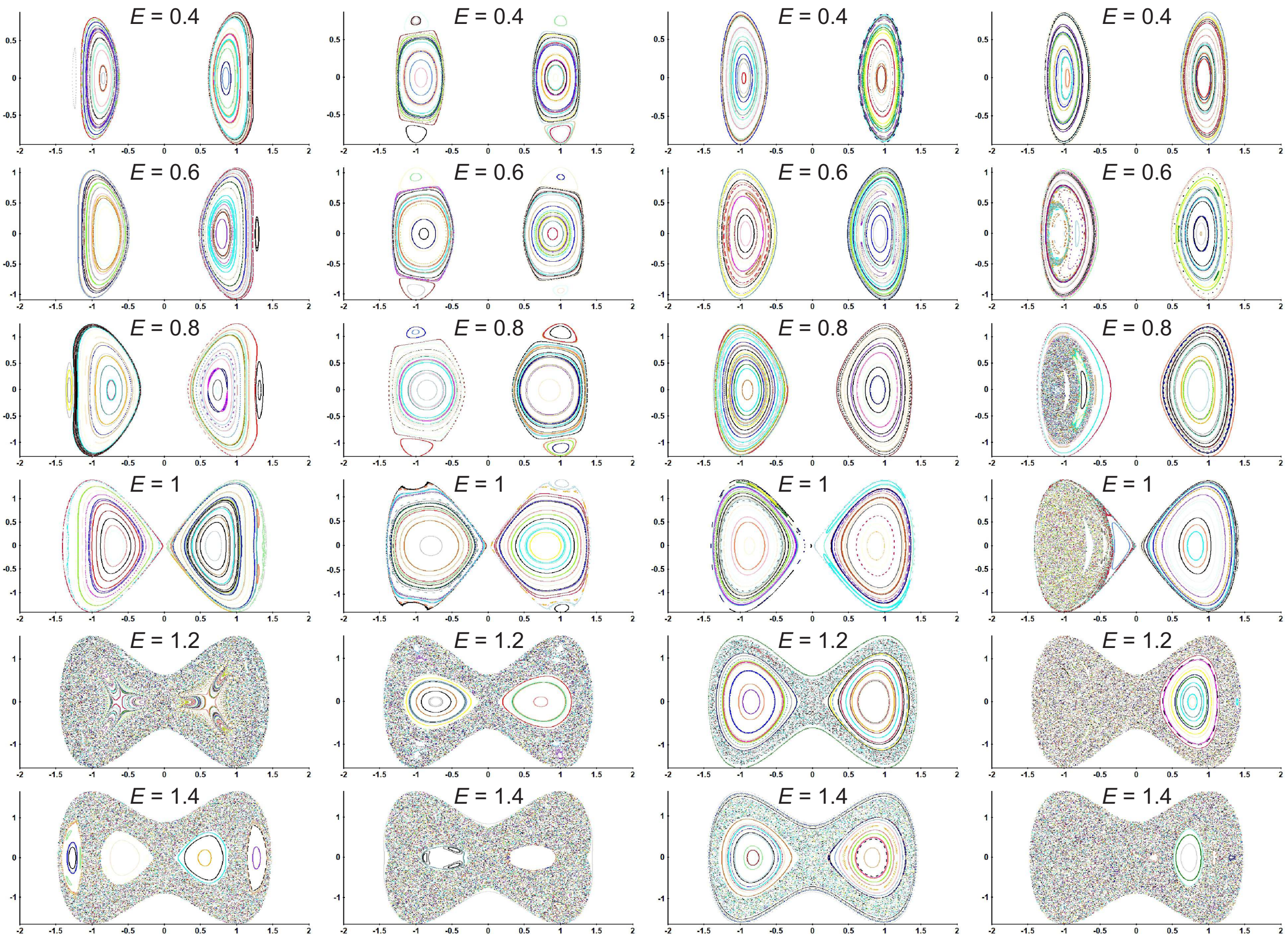
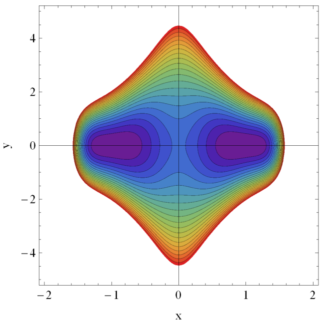
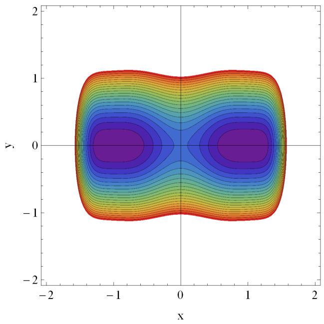


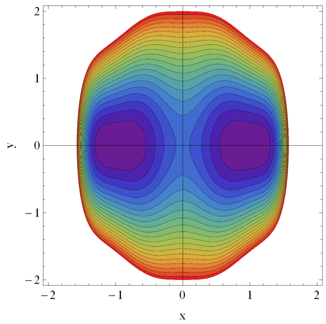
Figure 1: Properties of the Hamiltonian (1) for $A = 0, \kappa = 2$ and various values of the parameters B, C, μ (specified in the captions; default values are $B = 0, C = 1, \mu = 0.2$). The first row shows the equipotential surfaces, the second row shows the section $y = 0$ of the potential (note that this section is independent of the parameters B, C, μ). Below there are Poincaré sections for each configuration at 6 selected energies, which are marked by red lines in the figure of the second row. The outstretched minimum of the asymmetric $B = 0.8$ case is more chaotic than the squeezed minimum.



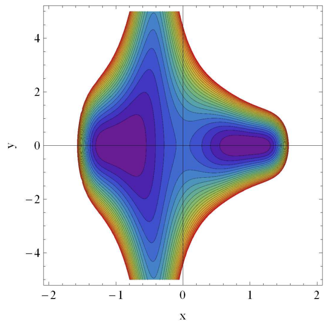
(a) $A = 0$



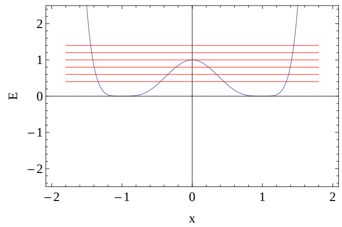
(b) $C = 3.8$



(c) $\mu = 1$



(d) $B = 0.8$



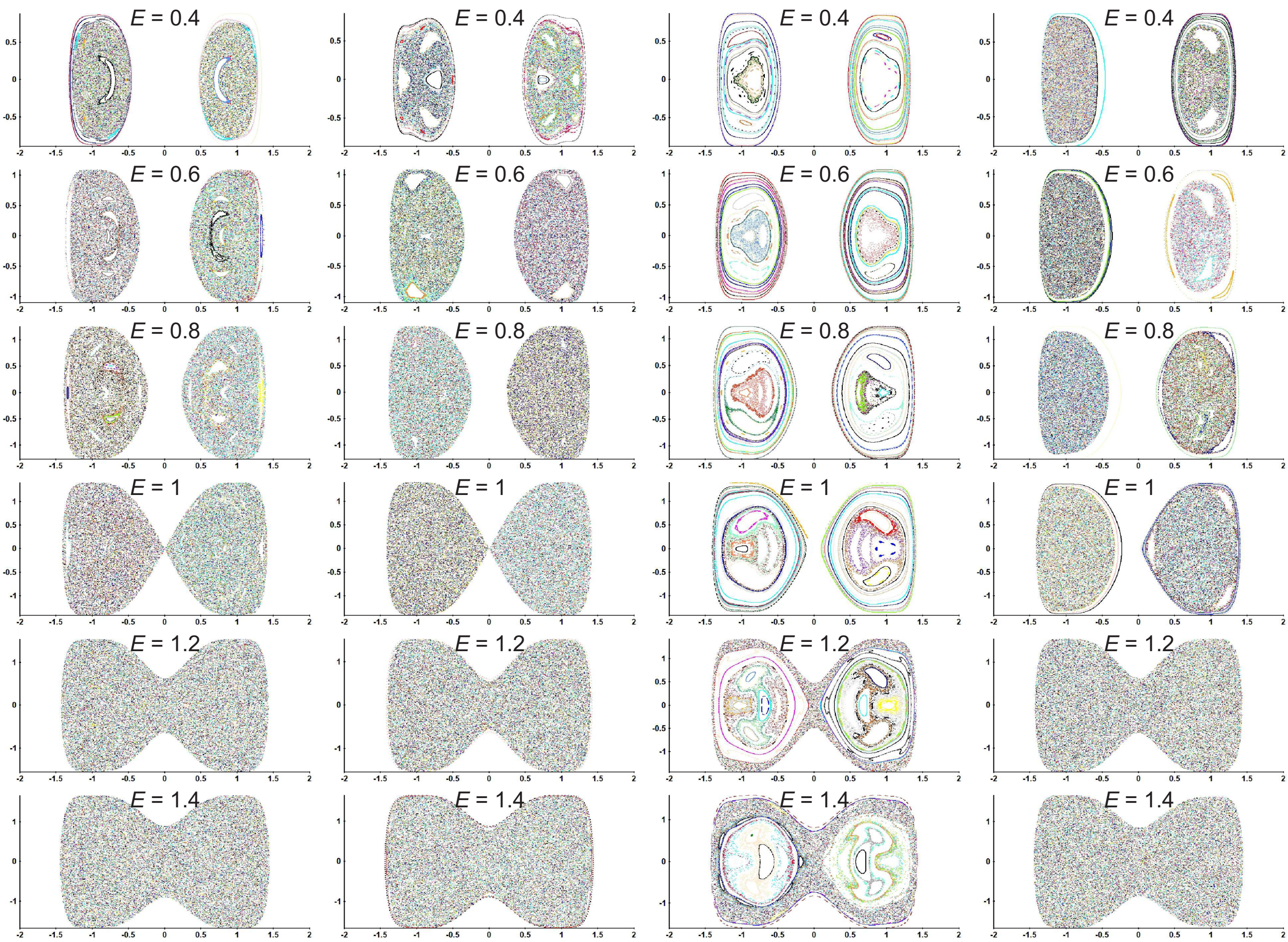


Figure 2: The same as in Fig. 1, but for the quartic case $\kappa = 4$. The system is more chaotic here.

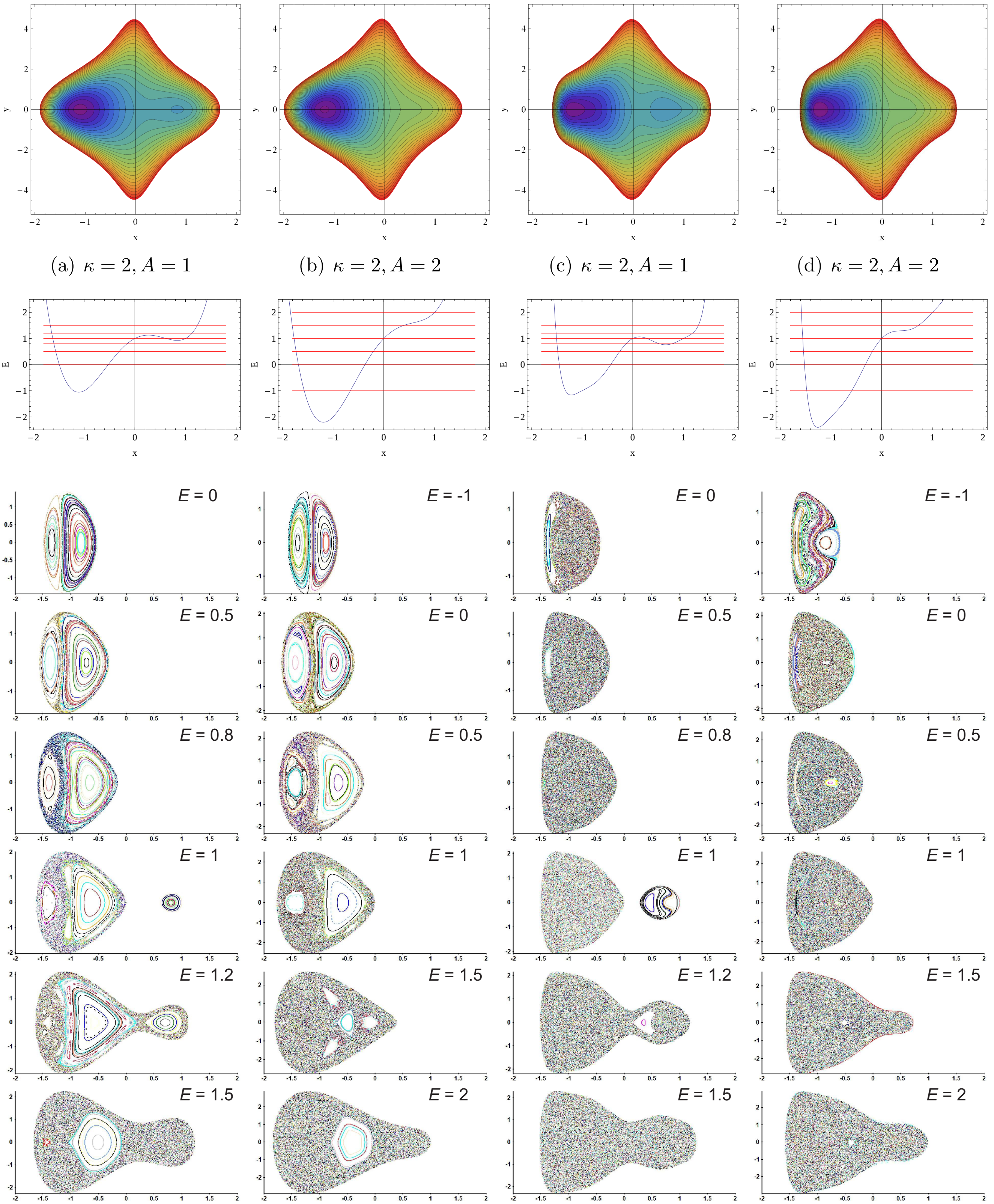


Figure 3: The same as in Fig. 1, but for varying the control parameter A. The first and second (third and fourth) column correspond with the quadratic $\kappa = 2$ (quartic $\kappa = 4$) case.

- In order to have a bound system, the condition

$$C + \mu > B \tag{5}$$

must be satisfied.

- If

$$C + \mu - B = 4, \tag{6}$$

the minimum on the left have the same frequencies in the x and y directions. If the left-hand side of the equation is smaller (higher) than 4, the minimum is γ-softer (more γ-rigid)[1]. The ratio of the frequencies is

$$R^{(-)} \equiv \frac{\Omega_y^-}{\Omega_x^-} = \frac{1}{2}\sqrt{C + \mu - B}. \tag{7}$$

- The same is true for the right minimum, but the equation is

$$C + \mu + B = 4, \tag{8}$$

and the rigidity ratio reads

$$R^{(+)} \equiv \frac{\Omega_y^+}{\Omega_x^+} = \frac{1}{2}\sqrt{C + \mu + B}. \tag{9}$$

- The frequencies in the x direction are the same in both minima.

By tuning the parameters B, C, μ, one can have all the possible configurations: (a) a soft-soft (small values of the parameters), (b) a rigid-rigid (large values of the parameters), (c) a symmetric-symmetric ($B = 0$, $C + \mu = 4$), (d) a symmetric-soft ($C + \mu + B = 4, 0 < C + \mu - B < 4$), (e) a symmetric-rigid ($C + \mu + B > 4, C + \mu - B = 4$), or (f) a soft-rigid (large values of the parameters, B slightly slower than the sum $C + \mu$). The evolution of the frequencies for three particular cases used later in the calculations of this notes are shown in Fig. 4 (note that the frequencies change, and hence the rigidity as well, when one moves out from the critical point $A = 0$).

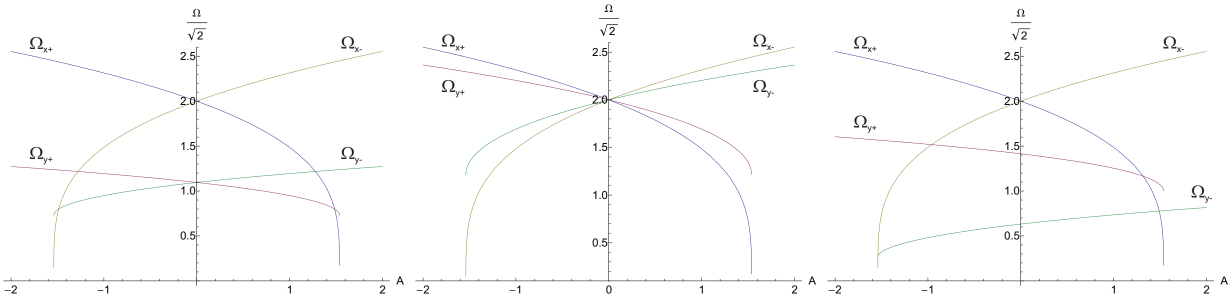
1.4 Rigidity for $\kappa = 4$

At the critical point, the potential (1) does not have the quadratic coefficient in the Taylor expansion at the minima. The reason is that the wells have the quartic shape in the x direction. Therefore, the lowest lying states form a mixture of a harmonic oscillator spectrum (coming from the y direction approximation) and a quartic oscillator spectrum (from the x direction approximation), made even more complicated by the effect of tunneling. Out of the critical point, however, the minima do have a quadratic approximation, given by a solution of an algebraic equation of the 8^{th} order (not presented here).

1.5 Quantum dynamics

The Creagh-Whelan system can be diagonalized in:

[1] In this text γ-soft is equivalent to y-soft. The name γ-soft is used in order to be consistent with the nuclear physics jargon.



(a) (b) $C = 3.8$ (c) $B = 0.8$

Figure 4: Dependence of the frequencies of the harmonic approximation in the x and y directions on A at both local minima, calculated in the quadratic case $\kappa = 2$ for various values of the parameters B, C, μ (default values are $B = 0, C = 1, \mu = 0.2$). (a) A symmetric γ-softer case. (b) A symmetric case with the same x and y frequencies. (c) An asymmetric γ-soft case, having the left minimum softer than the right one.

1.5.1 Cartesian basis

$|n\rangle|m\rangle$ of two one-dimensional harmonic oscillators in x and y directions

$$\begin{aligned} H_x^{(0)} &= \frac{1}{2}p_x^2 + c_x x^2, \\ H_y^{(0)} &= \frac{1}{2}p_y^2 + c_y y^2, \end{aligned} \tag{10}$$

where

$$\begin{aligned} \langle n|H_x^{(0)}|n\rangle &= \hbar\Omega_x\left(n + \frac{1}{2}\right), \\ \langle m|H_y^{(0)}|m\rangle &= \hbar\Omega_y\left(m + \frac{1}{2}\right), \end{aligned} \tag{11}$$

and $\Omega_{x,y} = \sqrt{2c_{x,y}}$.

Hamiltonian (1) can be expanded and expressed as

$$H^{(\kappa=2)} = H_x^{(0)} + H_y^{(0)} + x^4 - (2 + c_x))\, x^2 + (\mu - c_y)\, y^2 + Ax + Bxy^2 + Cx^2y^2, \tag{12}$$

$$H^{(\kappa=4)} = H_x^{(0)} + H_y^{(0)} + x^8 - 4x^6 + 6x^4 - (4 + c_x))\, x^2 + (\mu - c_y)\, y^2 + Ax + Bxy^2 + Cx^2y^2. \tag{13}$$

Using the ladder operators

$$x = \sqrt{\frac{\hbar}{2\Omega_x}}\left(a + a^\dagger\right) = \frac{1}{s_x}\left(a + a^\dagger\right), \tag{14}$$

all the relevant nonzero matrix elements are

$$\begin{aligned}
\langle n|x|n+1\rangle &= \frac{1}{s_x}\sqrt{n+1} \\
\langle n|x^2|n\rangle &= \frac{1}{s_x^2}(2n+1) \\
\langle n|x^2|n+2\rangle &= \frac{1}{s_x^2}\sqrt{(n+2)(n+1)} \\
\langle n|x^4|n\rangle &= \frac{1}{s_x^4}\left[6n(n-1)+12n+3\right] \\
\langle n|x^4|n+2\rangle &= \frac{1}{s_x^4}(4n+6)\sqrt{(n+2)(n+1)} \\
\langle n|x^4|n+4\rangle &= \frac{1}{s_x^4}\sqrt{(n+4)(n+3)(n+2)(n+1)} \\
\langle n|x^6|n\rangle &= \frac{1}{s_x^6}\left[20n(n-1)(n-2)+90n(n-1)+90n+15\right] \\
\langle n|x^6|n+2\rangle &= \frac{1}{s_x^6}\left[15n(n-1)+60n+45\right]\sqrt{(n+2)(n+1)} \\
\langle n|x^6|n+4\rangle &= \frac{1}{s_x^6}(6n+15)\sqrt{(n+4)(n+3)(n+2)(n+1)} \\
\langle n|x^6|n+6\rangle &= \frac{1}{s_x^6}\sqrt{(n+6)(n+5)(n+4)(n+3)(n+2)(n+1)} \\
\langle n|x^8|n\rangle &= \frac{1}{s_x^8}\left[70n(n-1)(n-2)(n-3)+560n(n-1)(n-2)+1260n(n-1)+840n+105\right] \\
\langle n|x^8|n+2\rangle &= \frac{1}{s_x^8}\left[56n(n-1)(n-2)+420n(n-1)+840n+420\right]\sqrt{(n+2)(n+1)} \\
\langle n|x^8|n+4\rangle &= \frac{1}{s_x^8}\left[28n(n-1)+168n+210\right]\sqrt{(n+4)(n+3)(n+2)(n+1)} \\
\langle n|x^8|n+6\rangle &= \frac{1}{s_x^8}(8n+28)\sqrt{(n+6)(n+5)(n+4)(n+3)(n+2)(n+1)} \\
\langle n|x^8|n+8\rangle &= \frac{1}{s_x^8}\sqrt{(n+8)(n+7)(n+6)(n+5)(n+4)(n+3)(n+2)(n+1)}
\end{aligned} \tag{15}$$

and similarly for y and m.

The basis is cut in the following way: if we want to have N basis states, we find the value E_N such that the number of states $|n\rangle|m\rangle$ satisfying

$$\langle m|\langle n|\left(H_x^{(0)}+H_y^{(0)}\right)|n\rangle|m\rangle \leq E_N \tag{16}$$

is (in practice approximately) N. These states $|n\rangle|m\rangle$ are used for the diagonalization. The convergence depends on values of the adjustable parameters and is very poor in this basis. In general only a few percent of the lowest states converge.

1.5.2 Polar basis

$$|nm\rangle = R_{nm}(r)\Phi_m(\phi) \tag{17}$$

of the 2D harmonic oscillator

$$H_r^{(0)} = T + V_0 = -\frac{\hbar^2}{2}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right)+c_r r^2, \tag{18}$$

where

$$R_{nm}(r) = \sqrt{\frac{2k_r^2 n!}{(n+l)!}} \left(k_r^2 r^2\right)^{l/2} \mathrm{e}^{-\frac{1}{2}k_r^2 r^2} L_n^l \left(k_r^2 r^2\right) \tag{19}$$

$$\Phi_m(\phi) = \sqrt{\frac{1}{2\pi}} e^{\mathrm{i}m\phi} \tag{20}$$

(L_n^l denotes a Laguerre polynomial, $l = |m|$ and $k_r = \sqrt{\Omega_r/\hbar} = \sqrt[4]{2c_r/\hbar^2}$). The eigenvalues of (18) are

$$E_{nm}^{(0)} \equiv \langle nm|H_r^{(0)}|nm\rangle = \hbar\Omega_r(2n+l+1), \tag{21}$$

where $\Omega_r = \sqrt{2c_r}$.

The Hamiltonian (1), expressed in polar coordinates

$$x = r\cos\phi$$
$$y = r\sin\phi$$

reads

$$\begin{aligned} H^{(\kappa=2)} &= H_r^{(0)} + \frac{1}{8}r^4 \left[(1-C)\cos 4\phi + 4r^4\cos 2\phi + (3+C)\right] + \frac{1}{4}Br^3\left(\cos\phi - \cos 3\phi\right) \\ &\quad - r^2\cos 2\phi\left(\frac{\mu}{2}+1\right) + r^2\left(\frac{\mu}{2} - 1 - c_r\right) + Ar\cos\phi + 1, \\ H^{(\kappa=4)} &= H_r^{(0)} + \frac{1}{128}r^8\left(\cos 8\phi + 8\cos 6\phi + 28\cos 4\phi + 56\cos 2\phi + 35\right) \\ &\quad - \frac{1}{8}r^6\left(\cos 6\phi + 6\cos 4\phi + 15\cos 2\phi + 10\right) + \frac{1}{8}r^4\left[(6-C)\cos 4\phi + 24\cos 2\phi + (18+C)\right] \\ &\quad + \frac{1}{4}Br^3\left(\cos\phi - \cos 3\phi\right) - r^2\cos 2\phi\left(\frac{\mu}{2}+2\right) + r^2\left(\frac{\mu}{2} - 2 - c_r\right) + Ar\cos\phi + 1. \end{aligned} \tag{22}$$

1.5.3 Matrix elements

The general expression for the matrix elements is [2]

$$\begin{aligned} \langle n', m+b|\beta^a\cos b\gamma|nm\rangle &= \frac{1}{2}\left(1+\delta_{b0}\right)\sqrt{\frac{n'!n!}{(n'+l')!(n+l)!}}\frac{(-1)^{n'+n+a}}{k^{a/2}} \\ &\sum_s \frac{\left[\frac{1}{2}\left(a+l+l'\right)+s\right]!\left[\frac{1}{2}\left(a-\Delta l\right)\right]!\left[\frac{1}{2}\left(a+\Delta l\right)\right]!}{s!(n'-s)!(n-s)!\left[\frac{1}{2}(a-\Delta l)-n'+s\right]!\left[\frac{1}{2}(a+\Delta l)-n+s\right]!} \end{aligned} \tag{23}$$

where

$$l = |m| \qquad l' = |m'| \qquad \Delta l = l' - l \qquad \Delta m = |m' - m| \tag{24}$$

and the limits of the summation are

$$\begin{aligned} \max\left\{0, n' - \frac{1}{2}(a-\Delta l), n - \frac{1}{2}(a+\Delta l)\right\} \leq &s \leq \min\left\{n', n\right\} \qquad \text{for } a+b \text{ even,} \\ 0 \leq &s \leq \min\left\{n', n\right\} \qquad \text{for } a+b \text{ odd.} \end{aligned} \tag{25}$$

In our case $a + b$ is always even.

Explicit expressions of the necessary matrix elements for the Hamiltonian (1) are:

- $\Delta m = 0$:

$$\langle nl|r^2|nl\rangle = \frac{1}{k_r^2}(2n+l+1) \tag{26}$$
$$\langle n+1,l|r^2|nl\rangle = -\frac{1}{k_r^2}\sqrt{(n+1)(n+l+1)}$$

$$\langle nl|r^4|nl\rangle = \frac{1}{k_r^4}\left[n(n-1)+(m+l+1)(5n+l+2)\right] \tag{27}$$
$$\langle n+1,l|r^4|nl\rangle = -\frac{2}{k_r^4}\,(2n+l+2)\,\sqrt{(n+1)(n+l+1)}$$
$$\langle n+2,l|r^4|nl\rangle = \frac{1}{k_r^4}\sqrt{(n+1)(n+2)(n+l+1)(n+l+2)}$$

$$\begin{aligned}\langle nl|r^6|nl\rangle &= \frac{1}{k_r^6}[n(n-1)(n-2)+9n(n-1)(n+l+1)\\ &\quad+9n(n+l+1)(n+l+2)+(n+l+1)(n+l+2)(n+l+3)]\\ \langle n+1,l|r^6|nl\rangle &= -\frac{3}{k_r^6}[n(n-1)+3n(n+l+2)+(n+l+2)(n+l+3)]\sqrt{(n+1)(n+l+1)}\\ \langle n+2,l|r^6|nl\rangle &= \frac{3}{k_r^6}(2n+l+3)\sqrt{(n+1)(n+2)(n+l+1)(n+l+2)}\\ \langle n+3,l|r^6|nl\rangle &= -\frac{1}{k_r^6}\sqrt{(n+1)(n+2)(n+3)(n+l+1)(n+l+2)(n+l+3)}\end{aligned} \tag{28}$$

$$\begin{aligned}\langle nl|r^8|nl\rangle &= \frac{1}{k_r^8}[n(n-1)(n-2)(n-3)+16n(n-1)(n-2)(n+l+1)\\ &\quad+36n(n-1)(n+l+1)(n+l+2)+16n(n+l+1)(n+l+2)(n+l+3)\\ &\quad+(n+l+1)\cdots(n+l+4)]\\ \langle n+1,l|r^8|nl\rangle &= -\frac{4}{k_r^8}[n(n-1)(n-2)+6n(n-1)(n+l+2)+6n(n+l+2)(n+l+3)\\ &\quad+(n+l+2)(n+l+3)(n+l+4)]\sqrt{(n+1)(n+l+1)}\\ \langle n+2,l|r^8|nl\rangle &= \frac{2}{k_r^8}[3n(n-1)+8n(n+l+3)+3(n+l+3)(n+l+4)]\\ &\quad\sqrt{(n+1)(n+2)(n+l+1)(n+l+2)}\\ \langle n+3,l|r^8|nl\rangle &= -\frac{4}{k_r^8}(2n+l+4)\sqrt{(n+1)(n+2)(n+3)(n+l+1)(n+l+2)(n+l+3)}\\ \langle n+4,l|r^8|nl\rangle &= \frac{4}{k_r^8}\sqrt{(n+1)\cdots(n+4)(n+l+1)\cdots(n+l+4)}\end{aligned} \tag{29}$$

- $\Delta m = 1$:

$$\langle n-1,l+1|r\cos\phi|nl\rangle = -\frac{1}{k_r}\sqrt{n} \tag{30}$$
$$\langle n,l+1|r\cos\phi|nl\rangle = \frac{1}{k_r}\sqrt{n+l+1}$$

$$\begin{aligned}
\langle n-2, m+1|r^3\cos\phi|nm\rangle &= \frac{1}{2k_r^3}\sqrt{n(n-1)(n+l)} \\
\langle n-1, m+1|r^3\cos\phi|nm\rangle &= -\frac{1}{2k_r^3}(3n+2l+1)\sqrt{n} \\
\langle n, m+1|r^3\cos\phi|nm\rangle &= \frac{1}{2k_r^3}(3n+l+2)\sqrt{n+l+1} \\
\langle n+1, m+1|r^3\cos\phi|nm\rangle &= -\frac{1}{2k_r^3}\sqrt{(n+1)(n+l+1)(n+l+2)}
\end{aligned} \tag{31}$$

- $\Delta m = 2$:

$$\begin{aligned}
\langle n-2, l+2|r^2\cos 2\phi|nl\rangle &= \frac{1}{2k_r^2}\sqrt{n(n-1)} \\
\langle n-1, l+2|r^2\cos 2\phi|nl\rangle &= -\frac{1}{k_r^2}\sqrt{n(n+l+1)} \\
\langle n, l+2|r^2\cos 2\phi|nl\rangle &= \frac{1}{2k_r^2}\sqrt{(n+l+1)(n+l+2)}
\end{aligned} \tag{32}$$

$$\begin{aligned}
\langle nl|r^2\cos 2\phi|nl\rangle &= \frac{1}{2}\langle nl|r^2|nl\rangle \\
\langle n+1, l|r^2\cos 2\phi|nl\rangle &= \frac{1}{2}\langle n+1, l|r^2|nl\rangle
\end{aligned}$$

$$\begin{aligned}
\langle n-3, l+2|r^4\cos 2\phi|nl\rangle &= -\frac{1}{2k_r^4}\sqrt{n(n-1)(n-2)(n+l)} \\
\langle n-2, l+2|r^4\cos 2\phi|nl\rangle &= \frac{1}{2k_r^4}(4n+3l+1)\sqrt{n(n-1)} \\
\langle n-1, l+2|r^4\cos 2\phi|nl\rangle &= -\frac{3}{2k_r^4}(2n+l+1)\sqrt{n(n+l+1)} \\
\langle n, l+2|r^4\cos 2\phi|nl\rangle &= \frac{1}{2k_r^4}(4n+l+3)\sqrt{(n+l+1)(n+l+2)} \\
\langle n+1, l+2|r^4\cos 2\phi|nl\rangle &= -\frac{1}{2k_r^4}\sqrt{(n+1)(n+l+1)(n+l+2)(n+l+3)}
\end{aligned} \tag{33}$$

$$\begin{aligned}
\langle nl|r^4\cos 2\phi|nl\rangle &= \frac{1}{2}\langle nl|r^4|nl\rangle \\
\langle n+1, l|r^4\cos 2\phi|nl\rangle &= \frac{1}{2}\langle n+1, l|r^4|nl\rangle \\
\langle n+2, l|r^4\cos 2\phi|nl\rangle &= \frac{1}{2}\langle n+2, l|r^4|nl\rangle
\end{aligned}$$

$$
\begin{aligned}
\langle n-4, l+2|r^6\cos 2\phi|nl\rangle &= \frac{1}{2k_r^6}\sqrt{n(n-1)(n-2)(n-3)(n+l)(n+l-1)} \\
\langle n-3, l+2|r^6\cos 2\phi|nl\rangle &= -\frac{1}{k_r^6}(3n+2l-1)\sqrt{n(n-1)(n-2)(n+l)} \\
\langle n-2, l+2|r^6\cos 2\phi|nl\rangle &= \frac{1}{2k_r^6}[(n-2)(n-3)+8(n-2)(n+l+1) \\
&\quad + 6(n+l+1)(n+l+2)]\sqrt{n(n-1)} \\
\langle n-1, l+2|r^6\cos 2\phi|nl\rangle &= -\frac{2}{k_r^6}[(n-1)(n-2)+3(n-1)(n+l+2) \\
&\quad + (n+l+2)(n+l+3)]\sqrt{n(n+l+1)} \\
\langle n, l+2|r^6\cos 2\phi|nl\rangle &= \frac{1}{2k_r^6}[6n(n-1)+8n(n+l+3) \\
&\quad + (n+l+3)(n+l+4)]\sqrt{(n+l+1)(n+l+2)} \\
\langle n+1, l+2|r^6\cos 2\phi|nl\rangle &= -\frac{1}{k_r^6}(3n+l+4)\sqrt{(n+1)(n+l+1)(n+l+2)(n+l+3)} \\
\langle n+2, l+2|r^6\cos 2\phi|nl\rangle &= \frac{1}{2k_r^6}\sqrt{(n+1)(n+2)(n+l+1)\cdots(n+l+4)}
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
\langle nl|r^6\cos 2\phi|nl\rangle &= \frac{1}{2}\langle nl|r^6|nl\rangle \\
\langle n+1, l|r^6\cos 2\phi|nl\rangle &= \frac{1}{2}\langle n+1, l|r^6|nl\rangle \\
\langle n+2, l|r^6\cos 2\phi|nl\rangle &= \frac{1}{2}\langle n+2, l|r^6|nl\rangle \\
\langle n+3, l|r^6\cos 2\phi|nl\rangle &= \frac{1}{2}\langle n+3, l|r^6|nl\rangle
\end{aligned}
$$

$$\begin{aligned}
\langle n-5,l+2|r^8\cos 2\phi|nl\rangle &= -\frac{1}{2k_r^8}\sqrt{n(n-1)\cdots(n-4)(n+l)(n+l-1)(n+l-2)} \\
\langle n-4,l+2|r^8\cos 2\phi|nl\rangle &= \frac{1}{2k_r^8}(8n+5l-7)\sqrt{n(n-1)(n-2)(n-3)(n+l)(n+l-1)} \\
\langle n-3,l+2|r^8\cos 2\phi|nl\rangle &= -\frac{1}{2k_r^8}[3(n-3)(n-4)+15(n-3)(n+l+1) \\
&\quad +10(n+l+1)(n+l+2)]\sqrt{n(n-1)(n-2)(n+l)} \\
\langle n-2,l+2|r^8\cos 2\phi|nl\rangle &= \frac{1}{2k_r^8}[(n-2)(n-3)(n-4)+15(n-2)(n-3)(n+l+1) \\
&\quad +30(n-2)(n+l+1)(n+l+2) \\
&\quad +10(n+l+1)(n+l+2)(n+l+3)]\sqrt{n(n-1)} \\
\langle n-1,l+2|r^8\cos 2\phi|nl\rangle &= -\frac{5}{2k_r^8}[(n-1)(n-2)(n-3)+6(n-1)(n-2)(n+l+2) \\
&\quad +6(n-1)(n+l+2)(n+l+3) \\
&\quad +(n+l+2)(n+l+3)(n+l+4)]\sqrt{n(n+l+1)} \\
\langle n,l+2|r^8\cos 2\phi|nl\rangle &= \frac{1}{2k_r^8}[10n(n-1)(n-2)+30n(n-1)(n+l+3) \\
&\quad +15n(n+l+3)(n+l+4)+(n+l+3)(n+l+4)(n+l+5)] \\
&\quad \sqrt{(n+l+1)(n+l+2)} \\
\langle n+1,l+2|r^8\cos 2\phi|nl\rangle &= -\frac{1}{2k_r^8}[10n(n-1)+15n(n+l+4)+3(n+l+4)(n+l+5)] \\
&\quad \sqrt{(n+1)(n+l+1)(n+l+2)(n+l+3)} \\
\langle n+2,l+2|r^8\cos 2\phi|nl\rangle &= \frac{1}{2k_r^8}(8n+3l+15) \\
&\quad \sqrt{(n+1)(n+2)(n+l+1)\cdots(n+l+4)} \\
\langle n+3,l+2|r^8\cos 2\phi|nl\rangle &= -\frac{1}{2k_r^8}\sqrt{(n+1)(n+2)(n+3)(n+l+1)\cdots(n+l+5)}
\end{aligned} \tag{35}$$

$$\begin{aligned}
\langle nl|r^8\cos 2\phi|nl\rangle &= \frac{1}{2}\langle nl|r^8|nl\rangle \\
\langle n+1,l|r^8\cos 2\phi|nl\rangle &= \frac{1}{2}\langle n+1,l|r^8|nl\rangle \\
\langle n+2,l|r^8\cos 2\phi|nl\rangle &= \frac{1}{2}\langle n+2,l|r^8|nl\rangle \\
\langle n+3,l|r^8\cos 2\phi|nl\rangle &= \frac{1}{2}\langle n+3,l|r^8|nl\rangle \\
\langle n+4,l|r^8\cos 2\phi|nl\rangle &= \frac{1}{2}\langle n+4,l|r^8|nl\rangle
\end{aligned}$$

- $\Delta m = 3$:

$$\begin{aligned}
\langle n-3, l+3|r^3\cos 3\phi|nl\rangle &= -\frac{1}{2k_r^3}\sqrt{n(n-1)(n-2)} \\
\langle n-2, l+3|r^3\cos 3\phi|nl\rangle &= \frac{3}{2k_r^3}\sqrt{n(n-1)(n+l+1)} \\
\langle n-1, l+3|r^3\cos 3\phi|nl\rangle &= -\frac{3}{2k_r^3}\sqrt{n(n+l+1)(n+l+2)} \\
\langle n, l+3|r^3\cos 3\phi|nl\rangle &= \frac{1}{2k_r^3}\sqrt{(n+l+1)(n+l+2)(n+l+3)}
\end{aligned} \tag{36}$$

$$\begin{aligned}
\langle n-2, l+1|r^3\cos 3\phi|nl\rangle &= \langle n-2, l+1|r^3\cos\phi|nm\rangle \\
\langle n-1, l+1|r^3\cos 3\phi|nl\rangle &= \langle n-1, l+1|r^3\cos\phi|nm\rangle \\
\langle n, l+1|r^3\cos 3\phi|nl\rangle &= \langle n, l+1|r^3\cos\phi|nm\rangle \\
\langle n+1, l+1|r^3\cos 3\phi|nl\rangle &= \langle n+1, l+1|r^3\cos\phi|nm\rangle
\end{aligned}$$

- $\Delta m = 4$:

$$\begin{aligned}
\langle n-4, l+4|r^4\cos 4\phi|nl\rangle &= \frac{1}{2k_r^4}\sqrt{n(n-1)(n-2)(n-3)} \\
\langle n-3, l+4|r^4\cos 4\phi|nl\rangle &= -\frac{2}{k_r^4}\sqrt{n(n-1)(n-2)(n+l+1)} \\
\langle n-2, l+4|r^4\cos 4\phi|nl\rangle &= \frac{3}{k_r^4}\sqrt{n(n-1)(n+l+1)(n+l+2)} \\
\langle n-1, l+4|r^4\cos 4\phi|nl\rangle &= -\frac{2}{k_r^4}\sqrt{n(n+l+1)(n+l+2)(n+l+3)} \\
\langle n, l+4|r^4\cos 4\phi|nl\rangle &= \frac{1}{2k_r^4}\sqrt{(n+l+1)\cdots(n+l+4)}
\end{aligned} \tag{37}$$

$$\begin{aligned}
\langle n-3, l+2|r^4\cos 4\phi|nl\rangle &= \langle n-3, l+2|r^4\cos 2\phi|nl\rangle \\
\langle n-2, l+2|r^4\cos 4\phi|nl\rangle &= \langle n-2, l+2|r^4\cos 2\phi|nl\rangle \\
\langle n-1, l+2|r^4\cos 4\phi|nl\rangle &= \langle n-1, l+2|r^4\cos 2\phi|nl\rangle \\
\langle n, l+2|r^4\cos 4\phi|nl\rangle &= \langle n, l+2|r^4\cos 2\phi|nl\rangle \\
\langle n+1, l+2|r^4\cos 4\phi|nl\rangle &= \langle n+1, l+2|r^4\cos 2\phi|nl\rangle
\end{aligned}$$

$$\begin{aligned}
\langle nl|r^4\cos 4\phi|nl\rangle &= \frac{1}{2}\langle nl|r^4|nl\rangle \\
\langle n+1, l|r^4\cos 4\phi|nl\rangle &= \frac{1}{2}\langle n+1, l|r^4|nl\rangle \\
\langle n+2, l|r^4\cos 4\phi|nl\rangle &= \frac{1}{2}\langle n+2, l|r^4|nl\rangle
\end{aligned}$$

$$
\begin{aligned}
\langle n-5,l+4|r^6\cos 4\phi|nl\rangle &= -\frac{1}{2k_r^6}\sqrt{n(n-1)\cdots(n-4)(n+l)} \\
\langle n-4,l+4|r^6\cos 4\phi|nl\rangle &= \frac{1}{2k_r^6}(6n+5l+1)\sqrt{n(n-1)(n-2)(n-3)} \\
\langle n-3,l+4|r^6\cos 4\phi|nl\rangle &= -\frac{5}{2k_r^6}(3n+2l+1)\sqrt{n(n-1)(n-2)(n+l+1)} \\
\langle n-2,l+4|r^6\cos 4\phi|nl\rangle &= \frac{5}{k_r^6}(2n+l+1)\sqrt{n(n-1)(n+l+1)(n+l+2)} \\
\langle n-1,l+4|r^6\cos 4\phi|nl\rangle &= -\frac{5}{2k_r^6}(3n+l+2)\sqrt{n(n+l+1)(n+l+2)(n+l+3)} \\
\langle n,l+4|r^6\cos 4\phi|nl\rangle &= \frac{1}{2k_r^6}(6n+l+5)\sqrt{(n+l+1)\cdots(n+l+4)} \\
\langle n+1,l+4|r^6\cos 4\phi|nl\rangle &= -\frac{1}{2k_r^6}\sqrt{(n+1)(n+l+1)\cdots(n+l+5)}
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
\langle n-4,l+2|r^6\cos 4\phi|nl\rangle &= \langle n-4,l+2|r^6\cos 2\phi|nl\rangle \\
\langle n-3,l+2|r^6\cos 4\phi|nl\rangle &= \langle n-3,l+2|r^6\cos 2\phi|nl\rangle \\
\langle n-2,l+2|r^6\cos 4\phi|nl\rangle &= \langle n-2,l+2|r^6\cos 2\phi|nl\rangle \\
\langle n-1,l+2|r^6\cos 4\phi|nl\rangle &= \langle n-1,l+2|r^6\cos 2\phi|nl\rangle \\
\langle n,l+2|r^6\cos 4\phi|nl\rangle &= \langle n,l+2|r^6\cos 2\phi|nl\rangle \\
\langle n+1,l+2|r^6\cos 4\phi|nl\rangle &= \langle n+1,l+2|r^6\cos 2\phi|nl\rangle \\
\langle n+2,l+2|r^6\cos 4\phi|nl\rangle &= \langle n+2,l+2|r^6\cos 2\phi|nl\rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle nl|r^6\cos 4\phi|nl\rangle &= \frac{1}{2}\langle nl|r^6|nl\rangle \\
\langle n+1,l|r^6\cos 4\phi|nl\rangle &= \frac{1}{2}\langle n+1,l|r^6|nl\rangle \\
\langle n+2,l|r^6\cos 4\phi|nl\rangle &= \frac{1}{2}\langle n+2,l|r^6|nl\rangle \\
\langle n+3,l|r^6\cos 4\phi|nl\rangle &= \frac{1}{2}\langle n+3,l|r^6|nl\rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle n-6,l+4|r^8\cos 4\phi|nl\rangle &= \frac{1}{2k_r^8}\sqrt{n(n-1)\cdots(n-5)(n+l)(n+l-1)} \\
\langle n-5,l+4|r^8\cos 4\phi|nl\rangle &= -\frac{1}{k_r^8}(4n+3l-2)\sqrt{n(n-1)\cdots(n-4)(n+l)} \\
\langle n-4,l+4|r^8\cos 4\phi|nl\rangle &= \frac{1}{2k_r^8}[(n-4)(n-5)+12(n-4)(n+l+1) \\
&\quad +15(n+l+1)(n+l+2)]\sqrt{n(n-1)(n-2)(n-3)} \\
\langle n-3,l+4|r^8\cos 4\phi|nl\rangle &= -\frac{1}{k_r^8}[3(n-3)(n-4)+15(n-3)(n+l+2) \\
&\quad +10(n+l+2)(n+l+3)]\sqrt{n(n-1)(n-2)(n+l+1)} \\
\langle n-2,l+4|r^8\cos 4\phi|nl\rangle &= \frac{5}{2k_r^8}[3(n-2)(n-3)+8(n-2)(n+l+3) \\
&\quad +3(n+l+3)(n+l+4)]\sqrt{n(n-1)(n+l+1)(n+l+2)} \\
\langle n-1,l+4|r^8\cos 4\phi|nl\rangle &= -\frac{1}{k_r^8}[10(n-1)(n-2)+15(n-1)(n+l+4) \\
&\quad +3(n+l+4)(n+l+5)]\sqrt{n(n+l+1)(n+l+2)(n+l+3)} \\
\langle n,l+4|r^8\cos 4\phi|nl\rangle &= \frac{1}{2k_r^8}[15n(n-1)+12n(n+l+5)+(n+l+5)(n+l+6)] \\
&\quad \sqrt{(n+l+1)\cdots(n+l+4)} \\
\langle n+1,l+4|r^8\cos 4\phi|nl\rangle &= -\frac{1}{k_r^8}(4n+l+6)\sqrt{(n+1)(n+l+1)\cdots(n+l+5)} \\
\langle n+2,l+4|r^8\cos 4\phi|nl\rangle &= \frac{1}{2k_r^8}\sqrt{(n+1)(n+2)(n+l+1)\cdots(n+l+6)}
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
\langle n-5,l+2|r^8\cos 4\phi|nl\rangle &= \langle n-5,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n-4,l+2|r^8\cos 4\phi|nl\rangle &= \langle n-4,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n-3,l+2|r^8\cos 4\phi|nl\rangle &= \langle n-3,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n-2,l+2|r^8\cos 4\phi|nl\rangle &= \langle n-2,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n-1,l+2|r^8\cos 4\phi|nl\rangle &= \langle n-1,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n,l+2|r^8\cos 4\phi|nl\rangle &= \langle n,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n+1,l+2|r^8\cos 4\phi|nl\rangle &= \langle n+1,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n+2,l+2|r^8\cos 4\phi|nl\rangle &= \langle n+2,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n+3,l+2|r^8\cos 4\phi|nl\rangle &= \langle n+3,l+2|r^8\cos 2\phi|nl\rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle nl|r^8\cos 4\phi|nl\rangle &= \frac{1}{2}\langle nl|r^8|nl\rangle \\
\langle n+1,l|r^8\cos 4\phi|nl\rangle &= \frac{1}{2}\langle n+1,l|r^8|nl\rangle \\
\langle n+2,l|r^8\cos 4\phi|nl\rangle &= \frac{1}{2}\langle n+2,l|r^8|nl\rangle \\
\langle n+3,l|r^8\cos 4\phi|nl\rangle &= \frac{1}{2}\langle n+3,l|r^8|nl\rangle \\
\langle n+4,l|r^8\cos 4\phi|nl\rangle &= \frac{1}{2}\langle n+4,l|r^8|nl\rangle
\end{aligned}
$$

- $\Delta m = 6$:

$$\begin{aligned}
\langle n-6, l+6|r^6\cos 6\phi|nl\rangle &= \frac{1}{2k_r^6}\sqrt{n(n-1)\cdots(n-5)} \\
\langle n-5, l+6|r^6\cos 6\phi|nl\rangle &= -\frac{3}{k_r^6}\sqrt{n(n-1)\cdots(n-4)(n+l+1)} \\
\langle n-4, l+6|r^6\cos 6\phi|nl\rangle &= \frac{15}{2k_r^6}\sqrt{n(n-1)(n-2)(n-3)(n+l+1)(n+l+2)} \\
\langle n-3, l+6|r^6\cos 6\phi|nl\rangle &= -\frac{10}{k_r^6}\sqrt{n(n-1)(n-2)(n+l+1)(n+l+2)(n+l+3)} \\
\langle n-2, l+6|r^6\cos 6\phi|nl\rangle &= \frac{15}{2k_r^6}\sqrt{n(n-1)(n+l+1)\cdots(n+l+4)} \\
\langle n-1, l+6|r^6\cos 6\phi|nl\rangle &= -\frac{3}{k_r^6}\sqrt{n(n+l+1)\cdots(n+l+5)} \\
\langle n, l+6|r^6\cos 6\phi|nl\rangle &= \frac{1}{2k_r^6}\sqrt{(n+l+1)\cdots(n+l+6)}
\end{aligned} \tag{40}$$

$$\begin{aligned}
\langle n-5, l+4|r^6\cos 6\phi|nl\rangle &= \langle n-5, l+4|r^6\cos 4\phi|nl\rangle \\
\langle n-4, l+4|r^6\cos 6\phi|nl\rangle &= \langle n-4, l+4|r^6\cos 4\phi|nl\rangle \\
\langle n-3, l+4|r^6\cos 6\phi|nl\rangle &= \langle n-3, l+4|r^6\cos 4\phi|nl\rangle \\
\langle n-2, l+4|r^6\cos 6\phi|nl\rangle &= \langle n-2, l+4|r^6\cos 4\phi|nl\rangle \\
\langle n-1, l+4|r^6\cos 6\phi|nl\rangle &= \langle n-1, l+4|r^6\cos 4\phi|nl\rangle \\
\langle n, l+4|r^6\cos 6\phi|nl\rangle &= \langle n, l+4|r^6\cos 4\phi|nl\rangle \\
\langle n+1, l+4|r^6\cos 6\phi|nl\rangle &= \langle n+1, l+4|r^6\cos 4\phi|nl\rangle
\end{aligned}$$

$$\begin{aligned}
\langle n-4, l+2|r^6\cos 6\phi|nl\rangle &= \langle n-4, l+2|r^6\cos 2\phi|nl\rangle \\
\langle n-3, l+2|r^6\cos 6\phi|nl\rangle &= \langle n-3, l+2|r^6\cos 2\phi|nl\rangle \\
\langle n-2, l+2|r^6\cos 6\phi|nl\rangle &= \langle n-2, l+2|r^6\cos 2\phi|nl\rangle \\
\langle n-1, l+2|r^6\cos 6\phi|nl\rangle &= \langle n-1, l+2|r^6\cos 2\phi|nl\rangle \\
\langle n, l+2|r^6\cos 6\phi|nl\rangle &= \langle n, l+2|r^6\cos 2\phi|nl\rangle \\
\langle n+1, l+2|r^6\cos 6\phi|nl\rangle &= \langle n+1, l+2|r^6\cos 2\phi|nl\rangle \\
\langle n+2, l+2|r^6\cos 6\phi|nl\rangle &= \langle n+2, l+2|r^6\cos 2\phi|nl\rangle
\end{aligned}$$

$$\begin{aligned}
\langle nl|r^6\cos 6\phi|nl\rangle &= \frac{1}{2}\langle nl|r^6|nl\rangle \\
\langle n+1, l|r^6\cos 6\phi|nl\rangle &= \frac{1}{2}\langle n+1, l|r^6|nl\rangle \\
\langle n+2, l|r^6\cos 6\phi|nl\rangle &= \frac{1}{2}\langle n+2, l|r^6|nl\rangle \\
\langle n+3, l|r^6\cos 6\phi|nl\rangle &= \frac{1}{2}\langle n+3, l|r^6|nl\rangle
\end{aligned}$$

$$\begin{aligned}
\langle n-7,l+6|r^8\cos 6\phi|nl\rangle &= -\frac{1}{2k_r^8}\sqrt{n(n-1)\cdots(n-6)(n+l)} \\
\langle n-6,l+6|r^8\cos 6\phi|nl\rangle &= \frac{1}{2k_r^8}(8n+7l+1)\sqrt{n(n-1)\cdots(n-5)} \\
\langle n-5,l+6|r^8\cos 6\phi|nl\rangle &= -\frac{7}{2k_r^8}(4n+3l+1)\sqrt{n(n-1)\cdots(n-4)(n+l+1)} \\
\langle n-4,l+6|r^8\cos 6\phi|nl\rangle &= \frac{7}{2k_r^8}(8n+5l+3)\sqrt{n(n-1)(n-2)(n-3)(n+l+1)(n+l+2)} \\
\langle n-3,l+6|r^8\cos 6\phi|nl\rangle &= -\frac{35}{2k_r^8}(2n+l+1)\sqrt{n(n-1)(n-2)(n+l+1)(n+l+2)(n+l+3)} \\
\langle n-2,l+6|r^8\cos 6\phi|nl\rangle &= \frac{7}{2k_r^8}(8n+3l+5)\sqrt{n(n-1)(n+l+1)\cdots(n+l+4)} \\
\langle n-1,l+6|r^8\cos 6\phi|nl\rangle &= -\frac{7}{2k_r^8}(4n+l+3)\sqrt{n(n+l+1)\cdots(n+l+5)} \\
\langle n,l+6|r^8\cos 6\phi|nl\rangle &= \frac{7}{2k_r^8}(8n+l+7)\sqrt{(n+l+1)\cdots(n+l+6)} \\
\langle n+1,l+6|r^8\cos 6\phi|nl\rangle &= -\frac{1}{2k_r^8}\sqrt{(n+1)(n+l+1)\cdots(n+l+7)}
\end{aligned} \tag{41}$$

$$\begin{aligned}
\langle n-6,l+4|r^8\cos 6\phi|nl\rangle &= \langle n-6,l+4|r^8\cos 4\phi|nl\rangle \\
\langle n-5,l+4|r^8\cos 6\phi|nl\rangle &= \langle n-5,l+4|r^8\cos 4\phi|nl\rangle \\
\langle n-4,l+4|r^8\cos 6\phi|nl\rangle &= \langle n-4,l+4|r^8\cos 4\phi|nl\rangle \\
\langle n-3,l+4|r^8\cos 6\phi|nl\rangle &= \langle n-3,l+4|r^8\cos 4\phi|nl\rangle \\
\langle n-2,l+4|r^8\cos 6\phi|nl\rangle &= \langle n-2,l+4|r^8\cos 4\phi|nl\rangle \\
\langle n-1,l+4|r^8\cos 6\phi|nl\rangle &= \langle n-1,l+4|r^8\cos 4\phi|nl\rangle \\
\langle n,l+4|r^8\cos 6\phi|nl\rangle &= \langle n,l+4|r^8\cos 4\phi|nl\rangle \\
\langle n+1,l+4|r^8\cos 6\phi|nl\rangle &= \langle n+1,l+4|r^8\cos 4\phi|nl\rangle \\
\langle n+2,l+4|r^8\cos 6\phi|nl\rangle &= \langle n+2,l+4|r^8\cos 4\phi|nl\rangle
\end{aligned}$$

$$\begin{aligned}
\langle n-5,l+2|r^8\cos 6\phi|nl\rangle &= \langle n-5,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n-4,l+2|r^8\cos 6\phi|nl\rangle &= \langle n-4,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n-3,l+2|r^8\cos 6\phi|nl\rangle &= \langle n-3,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n-2,l+2|r^8\cos 6\phi|nl\rangle &= \langle n-2,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n-1,l+2|r^8\cos 6\phi|nl\rangle &= \langle n-1,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n,l+2|r^8\cos 6\phi|nl\rangle &= \langle n,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n+1,l+2|r^8\cos 6\phi|nl\rangle &= \langle n+1,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n+2,l+2|r^8\cos 6\phi|nl\rangle &= \langle n+2,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n+3,l+2|r^8\cos 6\phi|nl\rangle &= \langle n+3,l+2|r^8\cos 2\phi|nl\rangle
\end{aligned}$$

$$
\begin{aligned}
\langle nl|r^8\cos 6\phi|nl\rangle &= \frac{1}{2}\langle nl|r^8|nl\rangle \\
\langle n+1,l|r^8\cos 6\phi|nl\rangle &= \frac{1}{2}\langle n+1,l|r^8|nl\rangle \\
\langle n+2,l|r^8\cos 6\phi|nl\rangle &= \frac{1}{2}\langle n+2,l|r^8|nl\rangle \\
\langle n+3,l|r^8\cos 6\phi|nl\rangle &= \frac{1}{2}\langle n+3,l|r^8|nl\rangle \\
\langle n+4,l|r^8\cos 6\phi|nl\rangle &= \frac{1}{2}\langle n+4,l|r^8|nl\rangle
\end{aligned}
$$

- $\Delta m = 8$:

$$
\begin{aligned}
\langle n-8,l+8|r^8\cos 8\phi|nl\rangle &= \frac{1}{2k_r^8}\sqrt{n(n-1)\cdots(n-7)} \qquad (42)\\
\langle n-7,l+8|r^8\cos 8\phi|nl\rangle &= -\frac{4}{k_r^8}\sqrt{n(n-1)\cdots(n-6)(n+l+1)} \\
\langle n-6,l+8|r^8\cos 8\phi|nl\rangle &= \frac{14}{k_r^8}\sqrt{n(n-1)\cdots(n-5)(n+l+1)(n+l+2)} \\
\langle n-5,l+8|r^8\cos 8\phi|nl\rangle &= -\frac{28}{k_r^8}\sqrt{n(n-1)\cdots(n-4)(n+l+1)(n+l+2)(n+l+3)} \\
\langle n-4,l+8|r^8\cos 8\phi|nl\rangle &= \frac{35}{k_r^8}\sqrt{n(n-1)(n-2)(n-3)(n+l+1)\cdots(n+l+4)} \\
\langle n-3,l+8|r^8\cos 8\phi|nl\rangle &= -\frac{28}{k_r^8}\sqrt{n(n-1)(n-2)(n+l+1)\cdots(n+l+5)} \\
\langle n-2,l+8|r^8\cos 8\phi|nl\rangle &= \frac{14}{k_r^8}\sqrt{n(n-1)(n+l+1)\cdots(n+l+6)} \\
\langle n-1,l+8|r^8\cos 8\phi|nl\rangle &= -\frac{4}{k_r^8}\sqrt{n(n+l+1)\cdots(n+l+7)} \\
\langle n,l+8|r^8\cos 8\phi|nl\rangle &= \frac{1}{2k_r^8}\sqrt{(n+l+1)\cdots(n+l+8)}
\end{aligned}
$$

$$
\begin{aligned}
\langle n-7,l+6|r^8\cos 8\phi|nl\rangle &= \langle n-7,l+6|r^8\cos 6\phi|nl\rangle \\
\langle n-6,l+6|r^8\cos 8\phi|nl\rangle &= \langle n-6,l+6|r^8\cos 6\phi|nl\rangle \\
\langle n-5,l+6|r^8\cos 8\phi|nl\rangle &= \langle n-5,l+6|r^8\cos 6\phi|nl\rangle \\
\langle n-4,l+6|r^8\cos 8\phi|nl\rangle &= \langle n-4,l+6|r^8\cos 6\phi|nl\rangle \\
\langle n-3,l+6|r^8\cos 8\phi|nl\rangle &= \langle n-3,l+6|r^8\cos 6\phi|nl\rangle \\
\langle n-2,l+6|r^8\cos 8\phi|nl\rangle &= \langle n-2,l+6|r^8\cos 6\phi|nl\rangle \\
\langle n-1,l+6|r^8\cos 8\phi|nl\rangle &= \langle n-1,l+6|r^8\cos 6\phi|nl\rangle \\
\langle n,l+6|r^8\cos 8\phi|nl\rangle &= \langle n,l+6|r^8\cos 6\phi|nl\rangle \\
\langle n+1,l+6|r^8\cos 8\phi|nl\rangle &= \langle n+1,l+6|r^8\cos 6\phi|nl\rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle n-6,l+4|r^8\cos 8\phi|nl\rangle &= \langle n-6,l+4|r^8\cos 4\phi|nl\rangle \\
\langle n-5,l+4|r^8\cos 8\phi|nl\rangle &= \langle n-5,l+4|r^8\cos 4\phi|nl\rangle \\
\langle n-4,l+4|r^8\cos 8\phi|nl\rangle &= \langle n-4,l+4|r^8\cos 4\phi|nl\rangle \\
\langle n-3,l+4|r^8\cos 8\phi|nl\rangle &= \langle n-3,l+4|r^8\cos 4\phi|nl\rangle \\
\langle n-2,l+4|r^8\cos 8\phi|nl\rangle &= \langle n-2,l+4|r^8\cos 4\phi|nl\rangle \\
\langle n-1,l+4|r^8\cos 8\phi|nl\rangle &= \langle n-1,l+4|r^8\cos 4\phi|nl\rangle \\
\langle n,l+4|r^8\cos 8\phi|nl\rangle &= \langle n,l+4|r^8\cos 4\phi|nl\rangle \\
\langle n+1,l+4|r^8\cos 8\phi|nl\rangle &= \langle n+1,l+4|r^8\cos 4\phi|nl\rangle \\
\langle n+2,l+4|r^8\cos 8\phi|nl\rangle &= \langle n+2,l+4|r^8\cos 4\phi|nl\rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle n-5,l+2|r^8\cos 8\phi|nl\rangle &= \langle n-5,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n-4,l+2|r^8\cos 8\phi|nl\rangle &= \langle n-4,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n-3,l+2|r^8\cos 8\phi|nl\rangle &= \langle n-3,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n-2,l+2|r^8\cos 8\phi|nl\rangle &= \langle n-2,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n-1,l+2|r^8\cos 8\phi|nl\rangle &= \langle n-1,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n,l+2|r^8\cos 8\phi|nl\rangle &= \langle n,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n+1,l+2|r^8\cos 8\phi|nl\rangle &= \langle n+1,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n+2,l+2|r^8\cos 8\phi|nl\rangle &= \langle n+2,l+2|r^8\cos 2\phi|nl\rangle \\
\langle n+3,l+2|r^8\cos 8\phi|nl\rangle &= \langle n+3,l+2|r^8\cos 2\phi|nl\rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle nl|r^8\cos 8\phi|nl\rangle &= \frac{1}{2}\langle nl|r^8|nl\rangle \\
\langle n+1,l|r^8\cos 8\phi|nl\rangle &= \frac{1}{2}\langle n+1,l|r^8|nl\rangle \\
\langle n+2,l|r^8\cos 8\phi|nl\rangle &= \frac{1}{2}\langle n+2,l|r^8|nl\rangle \\
\langle n+3,l|r^8\cos 8\phi|nl\rangle &= \frac{1}{2}\langle n+3,l|r^8|nl\rangle \\
\langle n+4,l|r^8\cos 8\phi|nl\rangle &= \frac{1}{2}\langle n+4,l|r^8|nl\rangle
\end{aligned}
$$

If the indices $i = \{n,m\}$, satisfying

$$2n + l + 1 \leq N, \tag{43}$$

of the Hamiltonian matrix are ordered in such a way that

$$
\begin{aligned}
&\ldots, i-1, i, i+1, \ldots \\
&\quad = \ldots, \{n-1,m\}, \{n,m\}, \{n+1,m\}, \ldots, \{n-1,m+1\}\{n,m+1\}, \{n+1,m+1\}, \ldots
\end{aligned} \tag{44}
$$

then the matrix has a symmetric band structure with only

$$W^{(\kappa=2)}(C=1) = 3(N+1)/2, \qquad W^{(\kappa=2)}(C\neq 1) = 2(N+1), \qquad W^{(\kappa=4)} = 4N \tag{45}$$

superdiagonals.

1.6 Level dynamics

The level dynamics for $\kappa = 2$ and $\kappa = 4$ and various values of the parameters B, C, μ is shown in Fig. 5 and 6, respectively. Figure 7 displays the same, but for a smaller value of the Planck constant (higher level density).

Signs of the "critical triangles" can be observed in the top part of Fig.5(b). Both minima are γ-soft, the left one is softer ($R^{(-)} \approx 0.32$) than the right one ($R^{(+)} =\approx 0.71$) (the left minimum has higher phase space volume for $A > 0$, therefore the level density is higher on the right of the graph).

2 Comparison with the CUSP

The 1D CUSP potential is

$$V = x^4 + bx^2 + ax + 1, \tag{46}$$

see Ref. [3] (note that the meaning of parameters a, b is swapped here). In order to compare easily the dynamics of this 1D system with the dynamics of the 2D Creagh-Whelan potential, the additive constant +1 is included and $b = -2$ chosen. The level dynamics is presented in Fig. 8 (note that the x dependence of the CUSP is the same as of the Creagh-Whelan potential $H^{(\kappa=2)}$).

3 Level density

The level density for the CUSP and Creagh-Whelan potential is shown in Fig. 9. It is calculated by Gaussian smoothing ($\sigma = 0.03$) of the level dynamics (CUSP: 2000 levels $b = -2, \hbar = 0.002$; Creagh-Whelan: 4000 levels $\kappa = 2, B = 0, C = 1, \mu = 0.2, \hbar = 0.03$). Sections of Fig. 9 at some energies and at some values of the control parameter are shown in Figs. 10 and 11, respectively.

It can be deduced from the figures that:

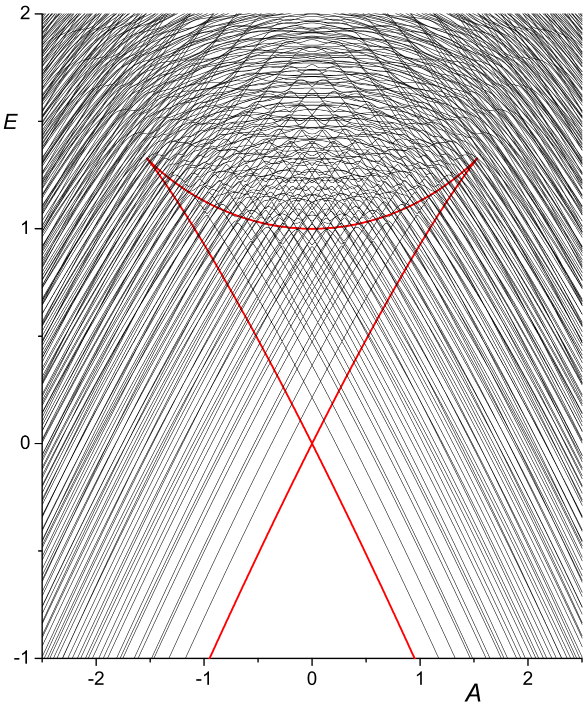
- CUSP has a jump in the level density on the lower legs of the critical triangle. Creagh-Whelan exhibits a jump in the first derivative of ρ (red dashed and green dotted line in Fig. 10).
- CUSP's level density diverges on the curved higher leg of the critical triangle (blue dash-dot line in Fig. 10) This divergence persists away from triangle in the direction of the top vertices, but it is smoothed out (cyan dash-dot-dot line in Fig. 10). Creagh-Whelan may have a divergence in the derivative of the level density (see $\mathrm{d}\rho/\mathrm{d}E$ of Fig. 11(b) and compare with ρ in Fig. 11(a)).

4 Flux of the levels

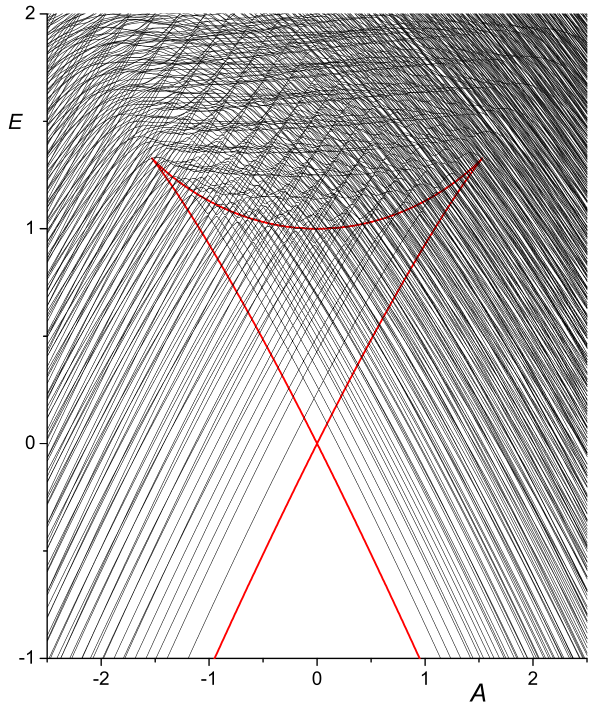
The flux is defined as the mean value

$$f = \left\langle \frac{\partial E(A)}{\partial A} \right\rangle \tag{47}$$

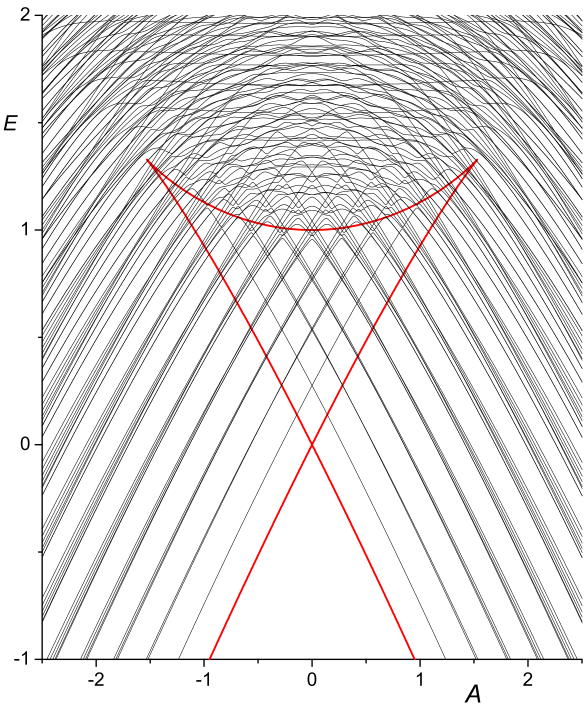
and its dispersion is denoted Δf. Here $E(A)$ are the energy levels from the level dynamics. In practice f is calculated by averaging $\partial E(A)/\partial A$ for each level on a small interval ΔA, while Δf is the dispersion. Then the values are sampled with a constant step in E direction and plotted in Fig. 12. The sections displayed in Figs. 13 and 14 are then obtained by cutting Fig. 12 along the black and white lines, respectively.



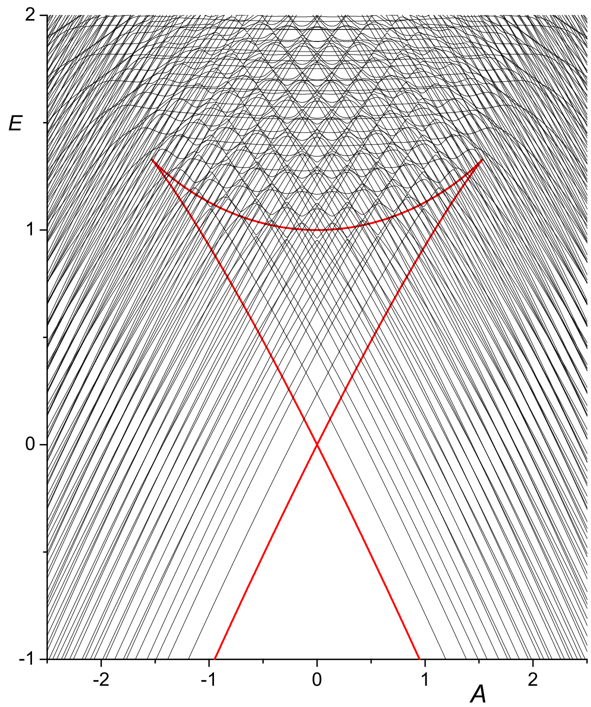
(a)



(b) $B = 0.8$

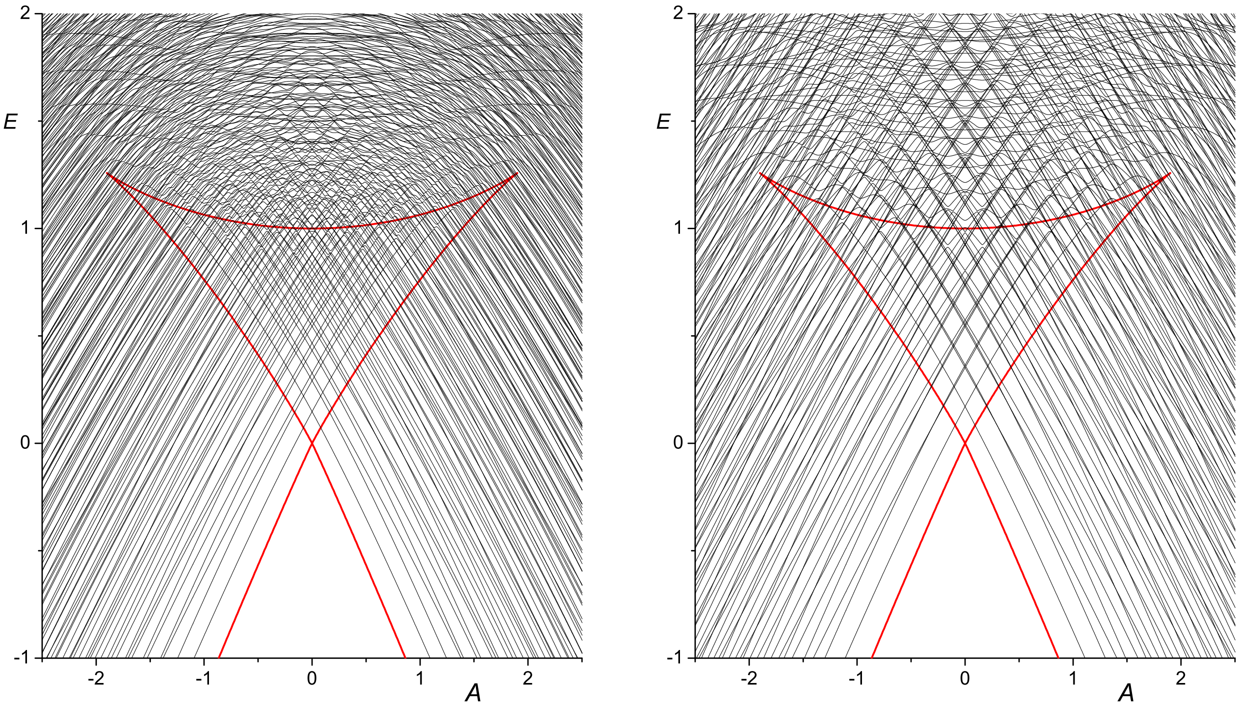


(c) $C = 3.8$



(d) $\mu = 1$

Figure 5: Level dynamics for the quadratic $\kappa = 2$ case with $\hbar = 0.1$. If not specified in the caption, the parameters are $B = 0, C = 1, \mu = 0.2$. Thick red lines repesent the position of the minima, and of the saddle point that separates them.



(a)

(b) $\mu = 1$

Figure 6: The same as in Fig. 5, but for the quartic $\kappa = 4$ case with $\hbar = 0.1$.

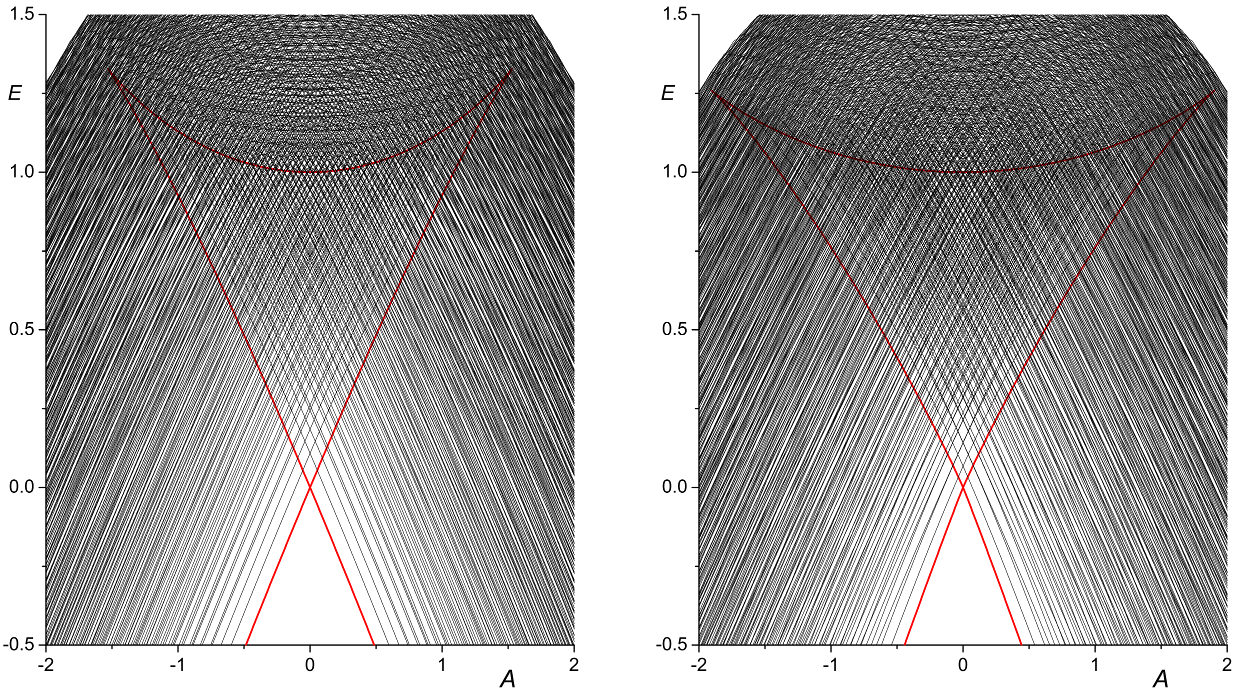


Figure 7: Level dynamics for (a) the quadratic $\kappa = 2$ case, and (b) the quartic $\kappa = 4$ case, with $B = 0, C = 1, \mu = 0.2, \hbar = 0.05$.

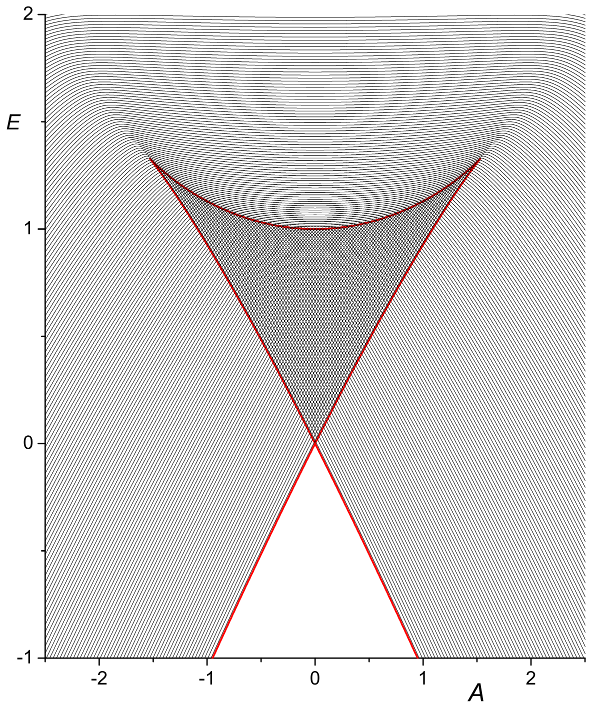


Figure 8: Level dynamics for CUSP potential (46) with $b = -2, \hbar = 0.01$.

It is observed that the derivative of the flux f has a step on the low vertices of the critical triange, the dispersion Δf has a step itself. The behaviour on the upper vertex is difficult to read[2].

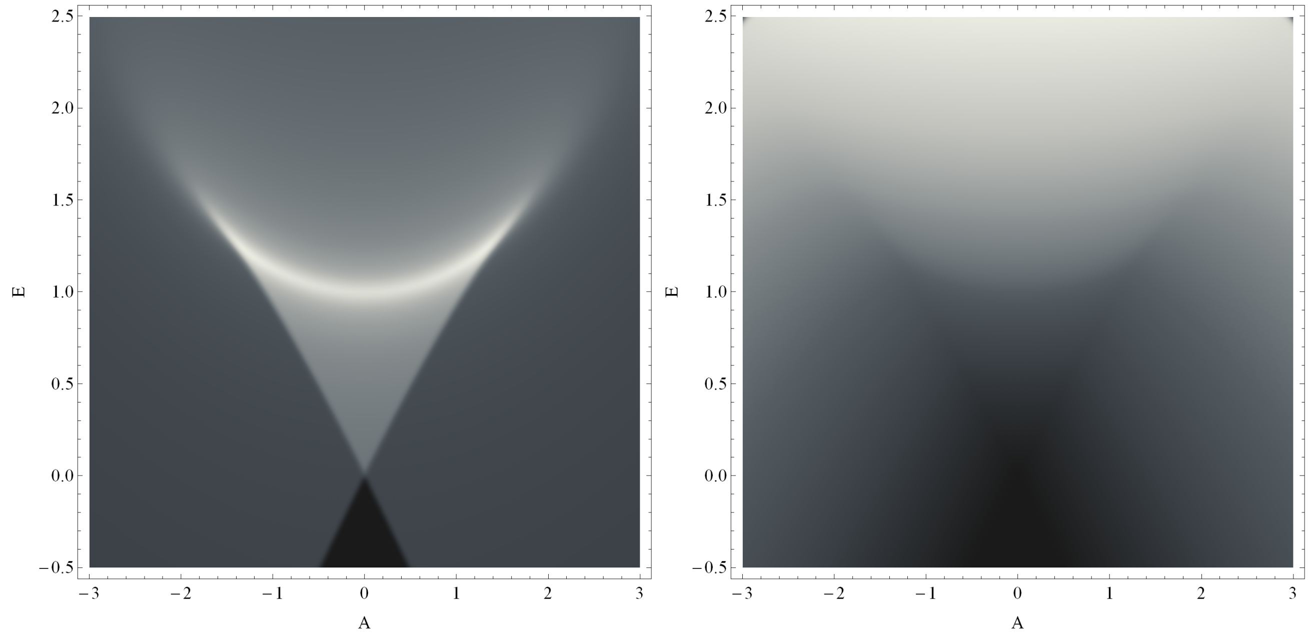
References

[1] S.C. Creagh and N.D. Whelan, *Phys. Rev. Lett.* **82**, 5237 (1999).

[2] S. Bell, *J. Phys. B: Atom. Molec. Phys.* **3**, 745 (1970).

[3] P. Cejnar and P. Stránský, *Phys. Rev.* E **78**, 031130 (2008).

[2]The precision of the calculation has almost reached the limit of my computer (the level dynamics is calculated for $\hbar = 0.03$, requiring 4000 well converging levels, step in A is 0.005, calculation time about several days). The graphical representation can be improved by a clever smoothing of the figures. (Do you have any idea how?)



(a) CUSP $b = -2$.

(b) Creagh-Whelan $\kappa = 2, B = 0, C = 1, \mu = 0.2$.

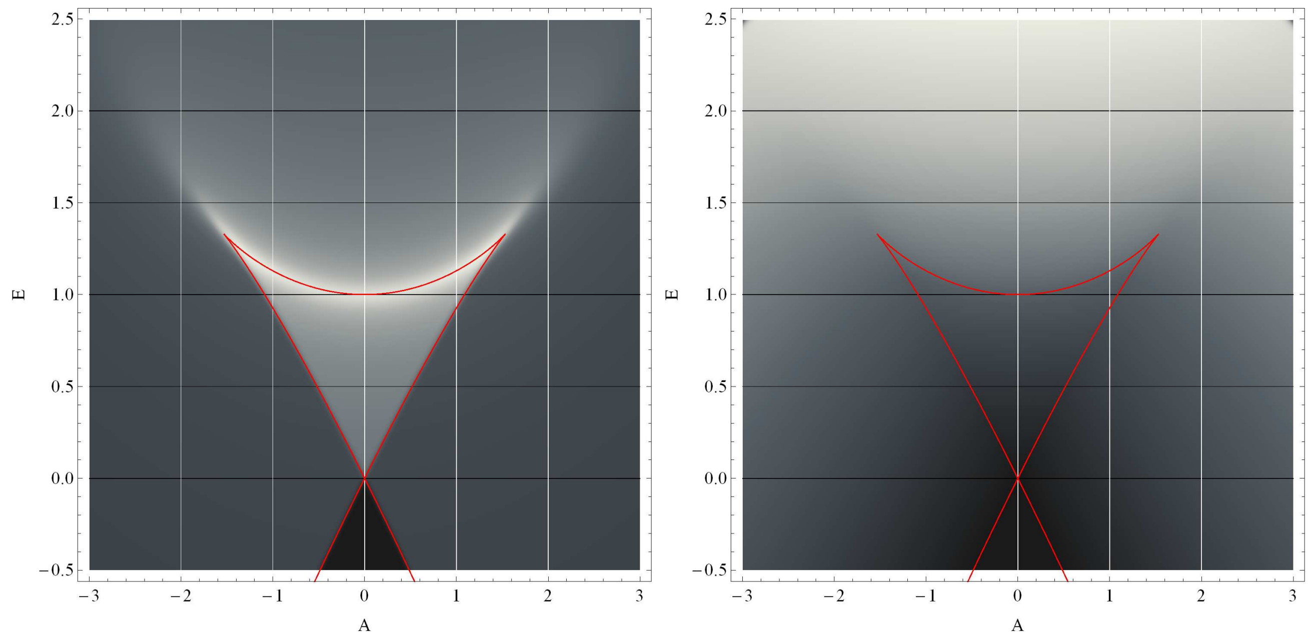
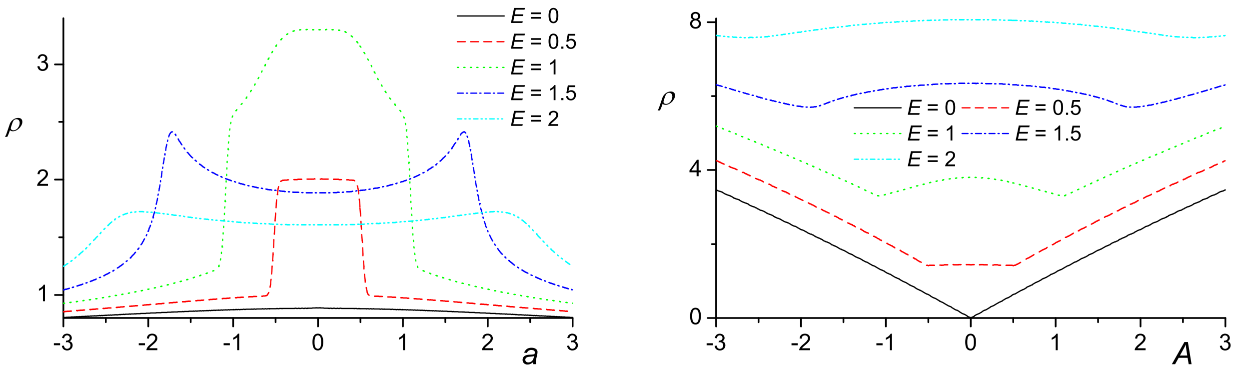


Figure 9: Level density. The second row includes the positions of the potential minima and of the saddle point (red lines), and the sections used in Fig. 10 (horizontal black lines) and in Fig. 11 (vertical white lines).



(a) CUSP

(b) Creagh-Whelan $\kappa = 2$.

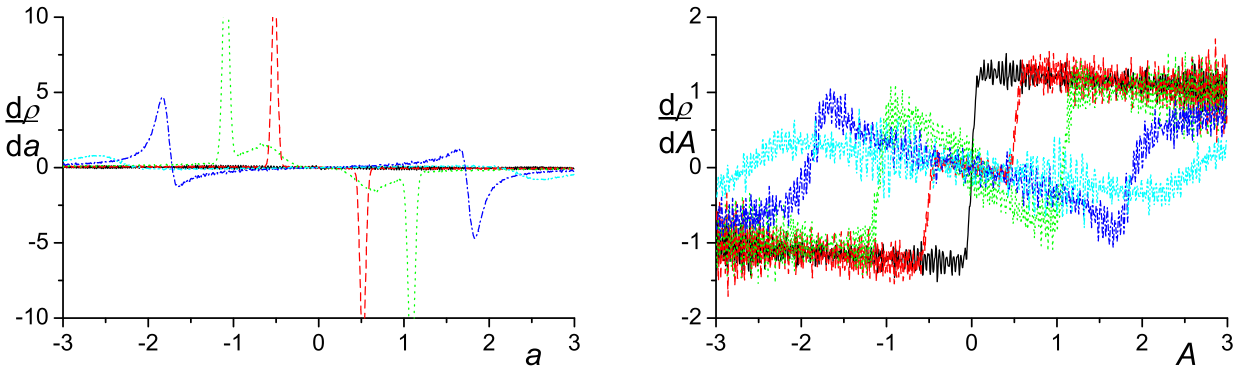
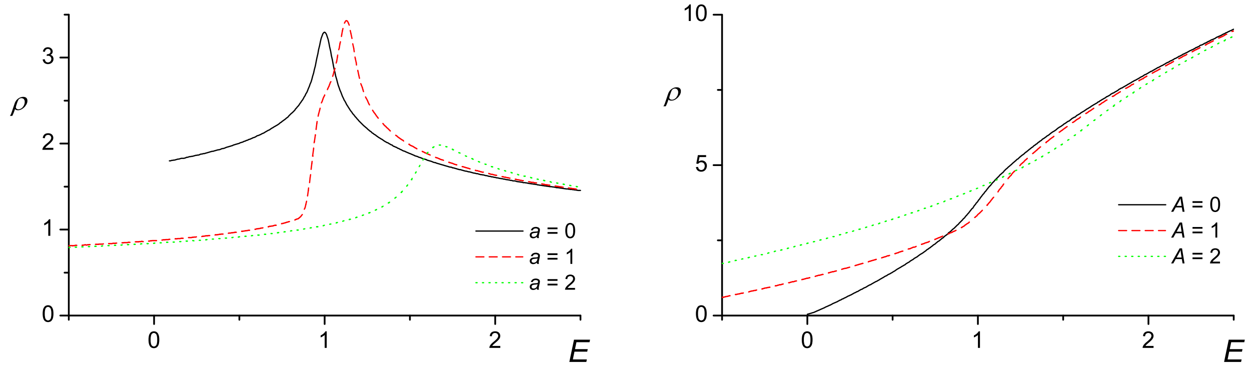


Figure 10: Level density (the first row) and its first derivative (the second row) at selected energies, which are marked by the horizonal black lines in Fig. 9.



(a) CUSP

(b) Creagh-Whelan $\kappa = 2$.

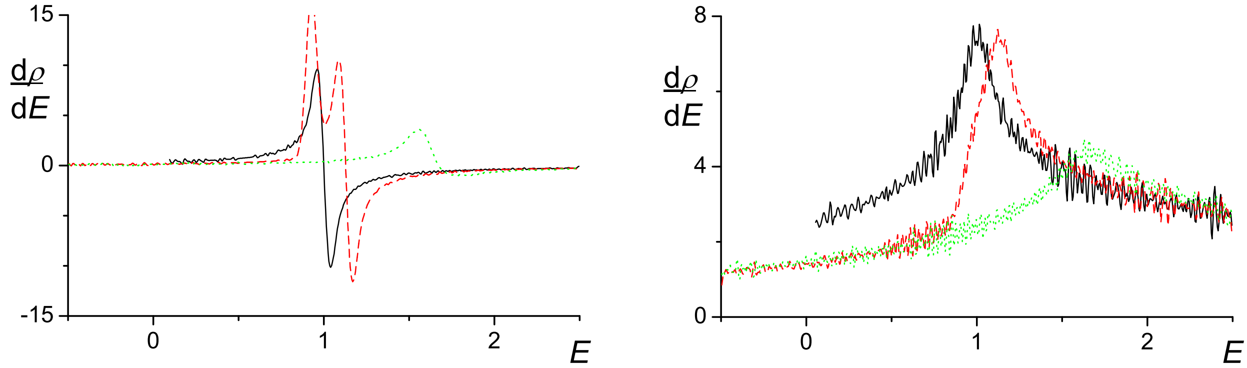


Figure 11: Level density (the first row) and its first derivative (the second row) at selected values of the control parameter a (CUSP) and A (Creagh-Whelan), which are marked by the vertical white lines in Fig. 9.

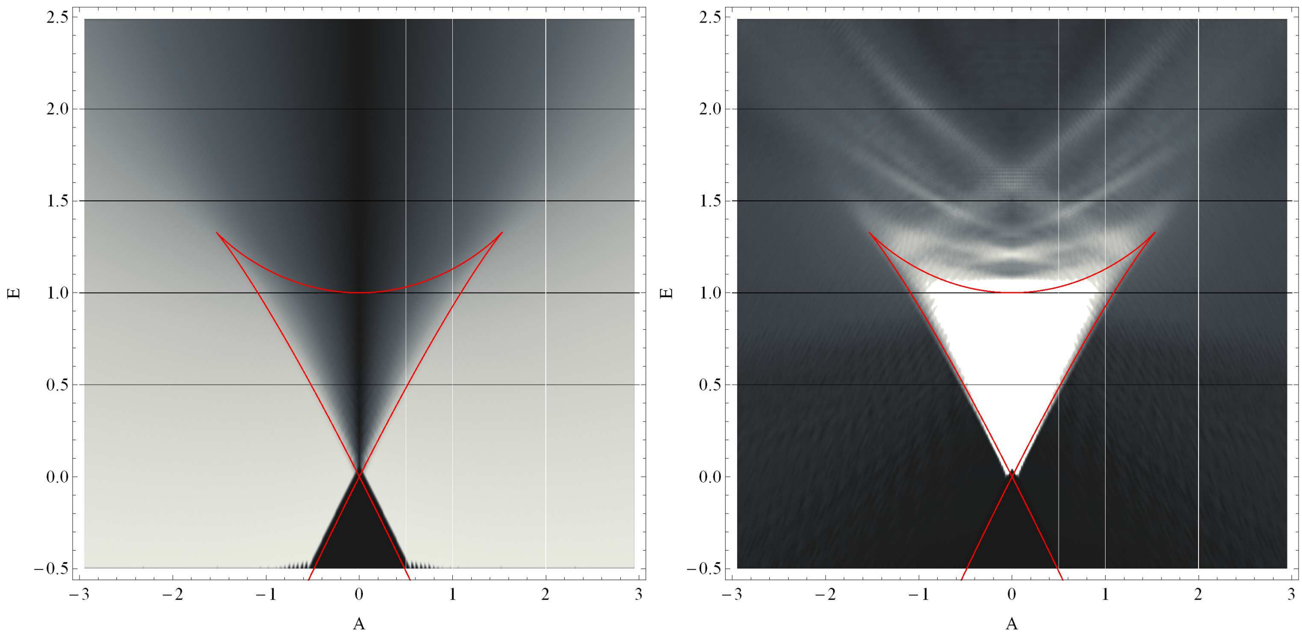


Figure 12: Flux and its dispersion of the Creagh-Whelan system with $B = 0, \mu = 0.2$, calculated with $hbar = 0.03$, 4000 well converging energy states. $\Delta A = 0.1$ (interval for the flux averaging and for calculation the dispersion). The positions of the minima and of the saddle point are plotted in red lines, the sections used in Figs. 13 and 14 and indicated by the horizontal black lines and the vertical white lines, respectively.

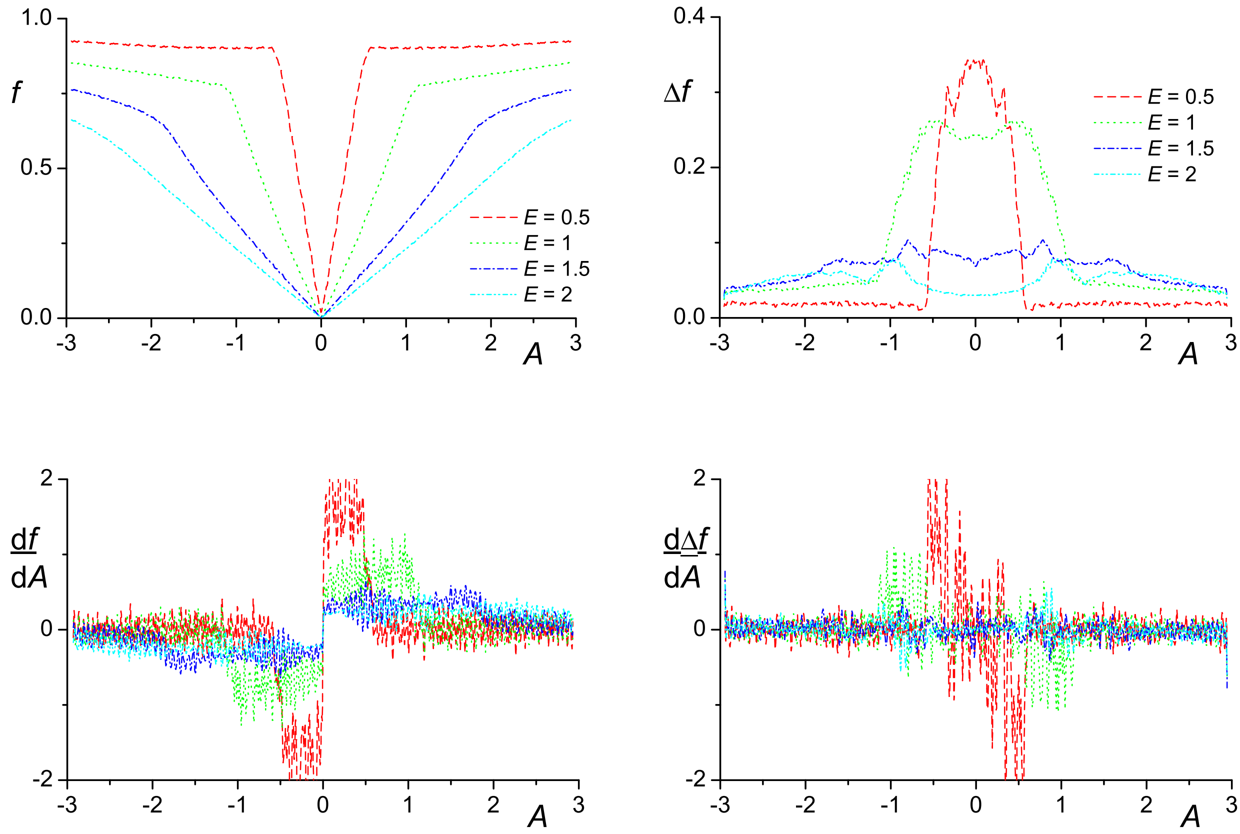


Figure 13: Flux and its dispersion (the first row) and their first derivative (the second row) for various energies E, marked by the horizontal black lines in Fig. 12.

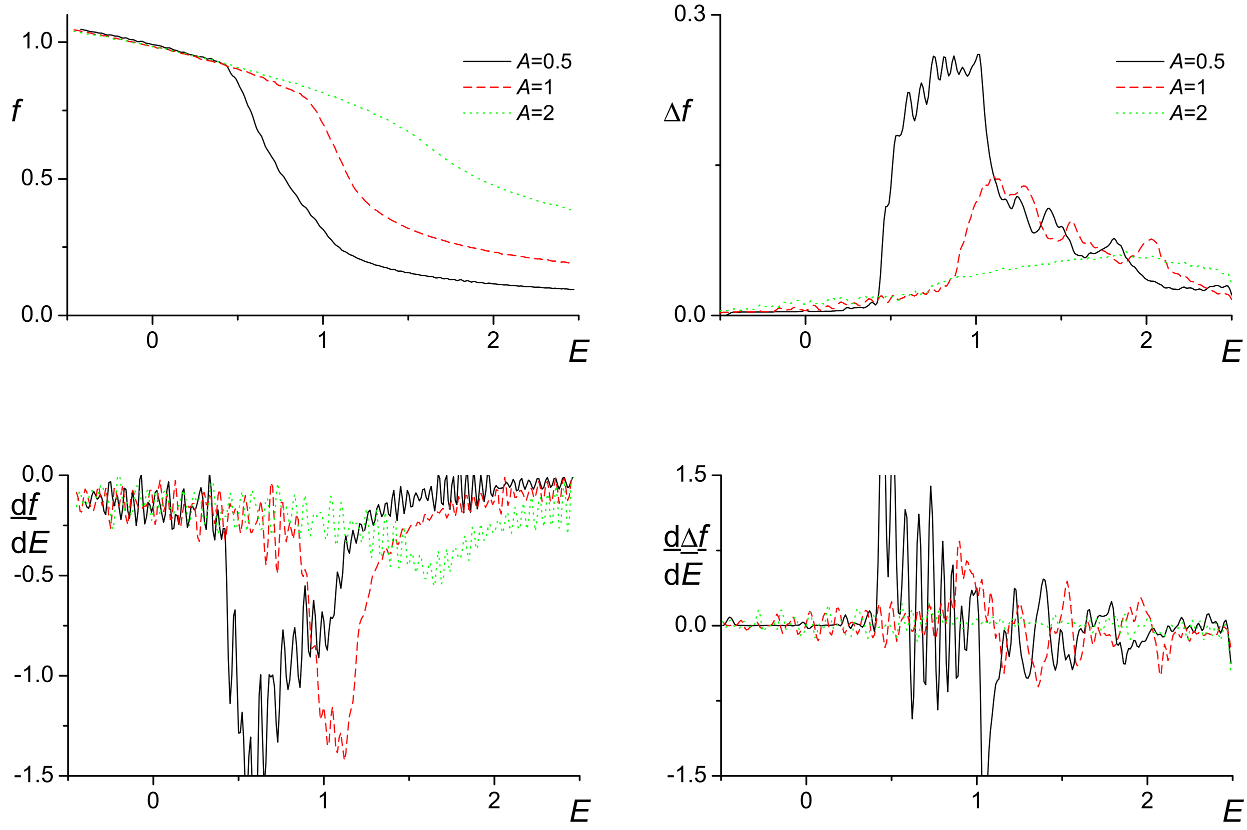


Figure 14: Flux and its dispersion (the first row) and their first derivative (the second row) at selected values of the control parameter A, marked by the vertical white lines in Fig. 9.